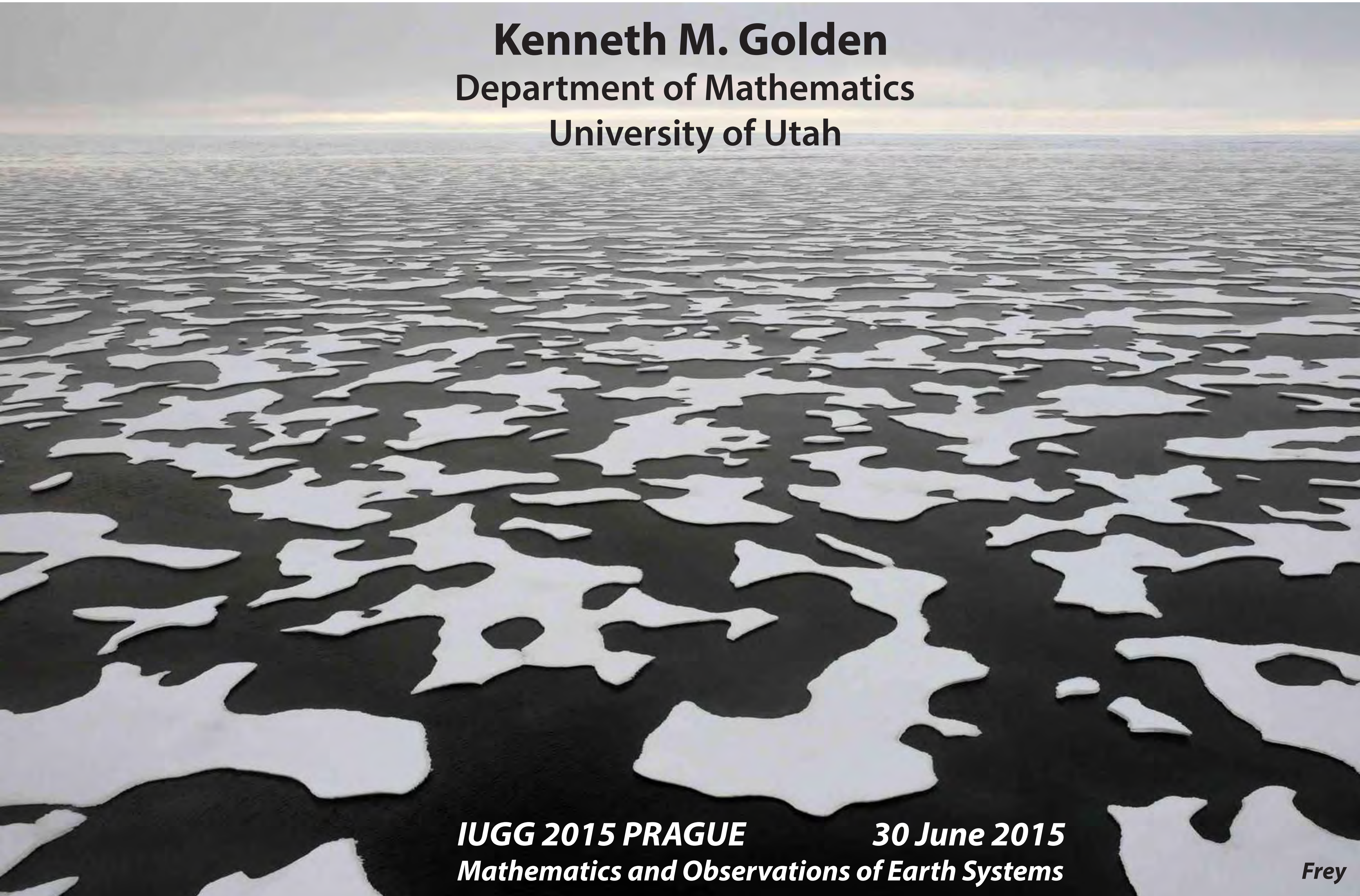
MODELING *the* MELT:

What math tells us about the disappearing polar ice caps

Kenneth M. Golden
Department of Mathematics
University of Utah



IUGG 2015 PRAGUE ***30 June 2015***

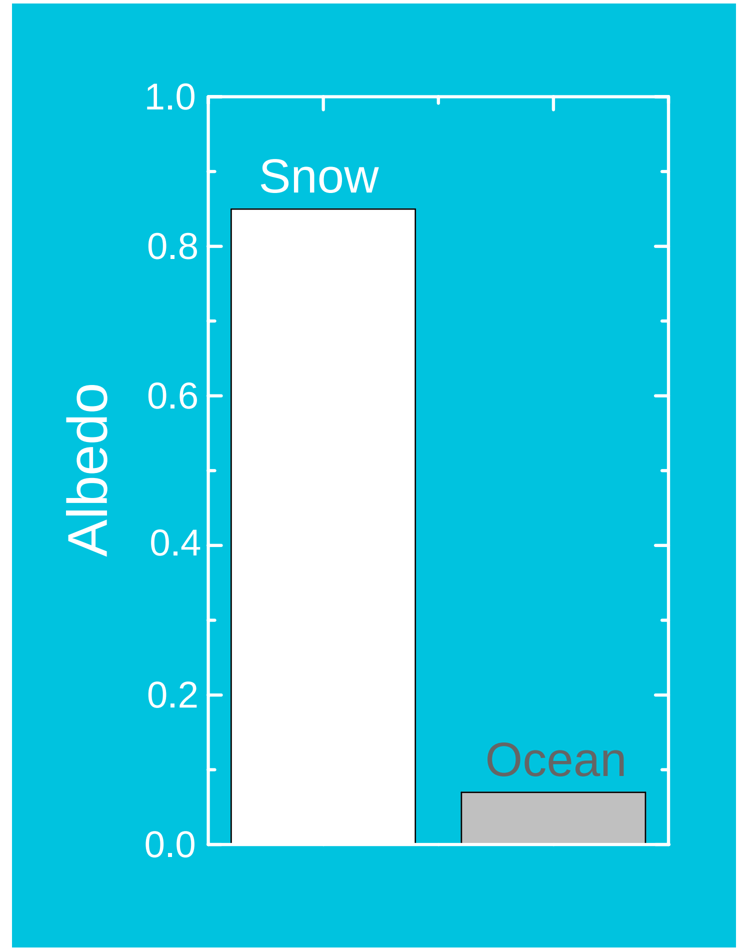
Mathematics and Observations of Earth Systems

Frey


SEA ICE covers 7 - 10% of earth's ocean surface
• boundary between ocean and atmosphere
• mediates exchange of heat, gases, momentum
• global ocean circulation
• indicator and agent of climate change

polar ice caps critical to global climate in reflecting incoming solar radiation

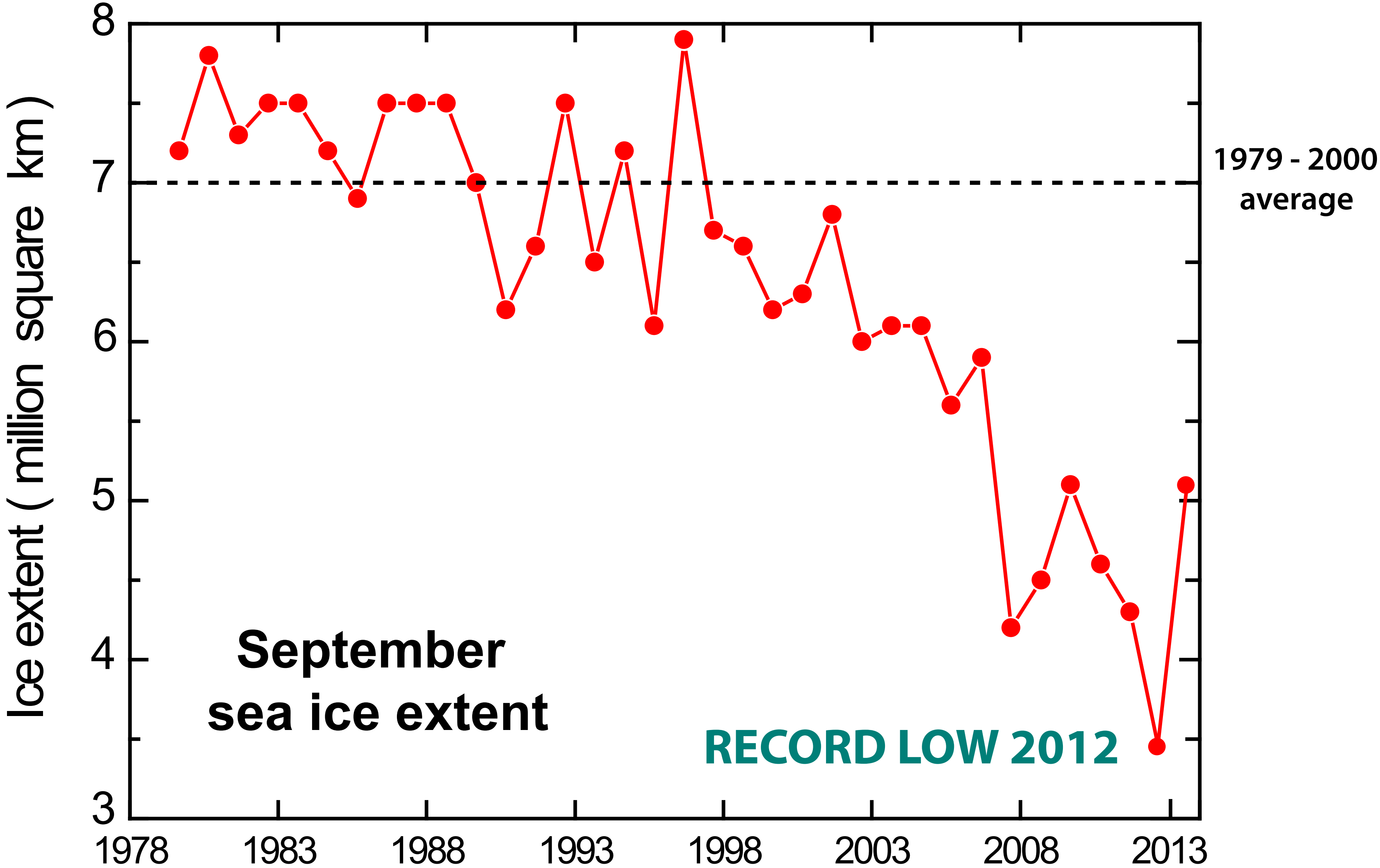
white snow and ice reflect



dark water and land absorb

$$\textit{\textbf{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

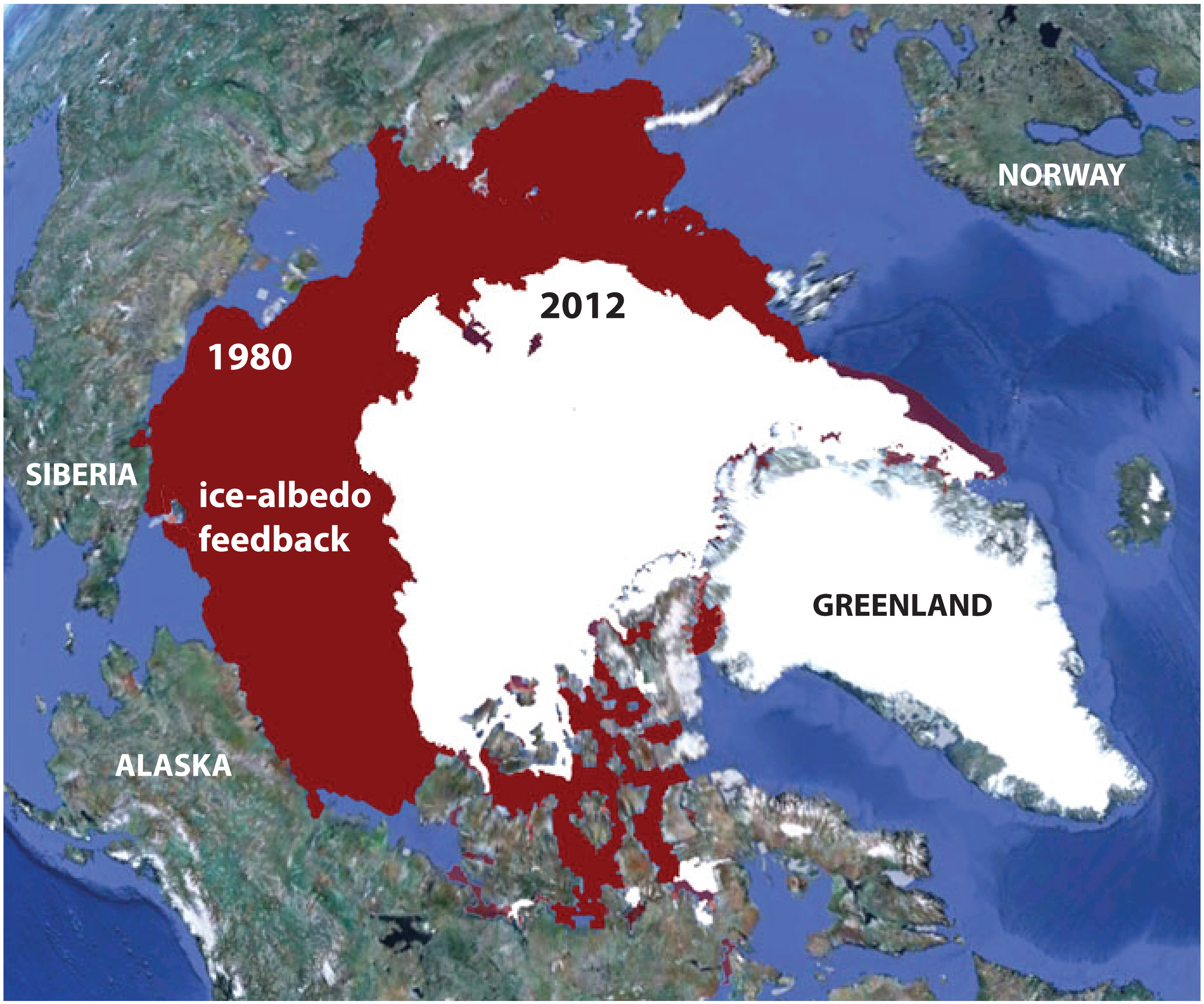
the summer Arctic sea ice pack is melting



National Snow and Ice Data Center

Change in Arctic Sea Ice Extent

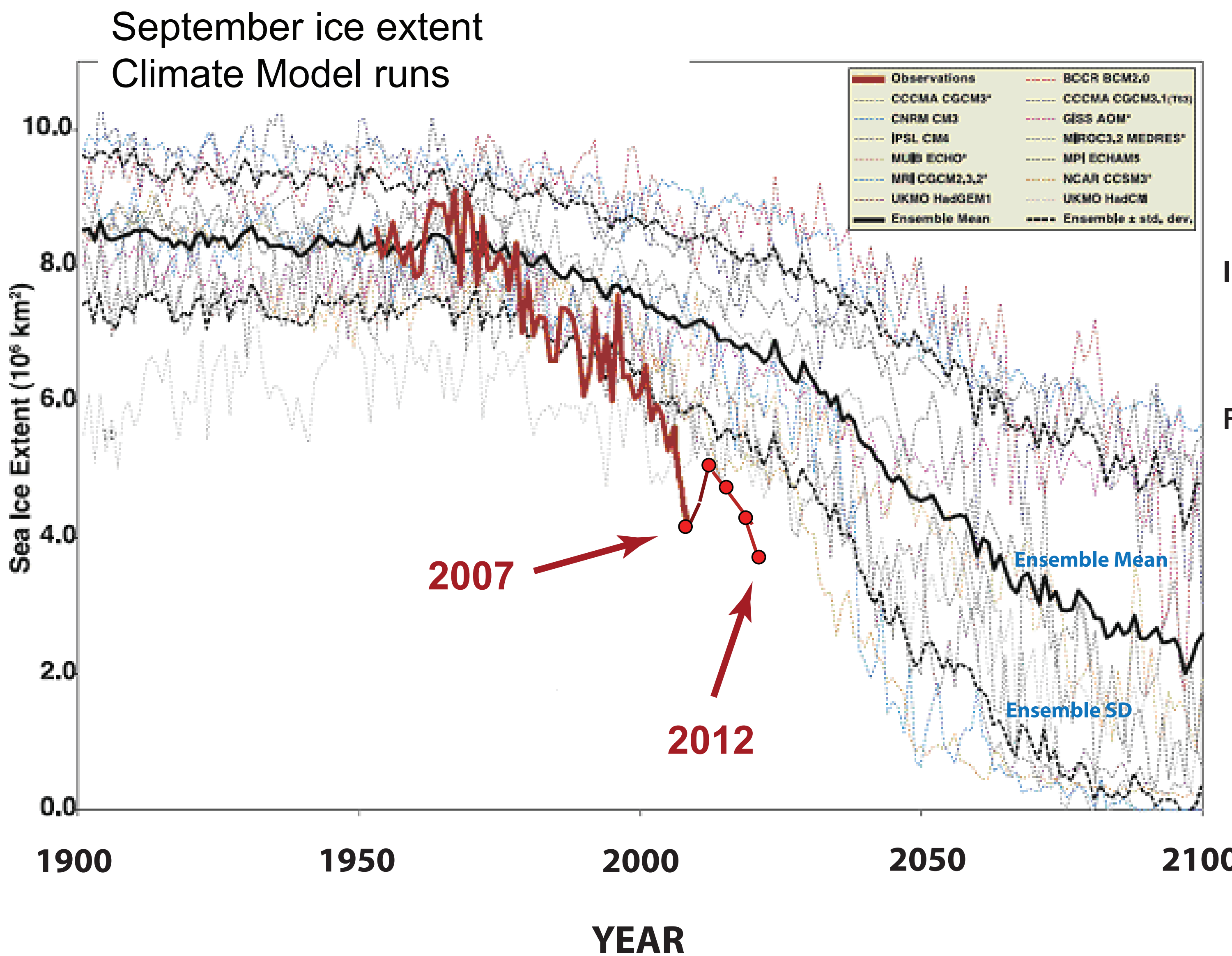
September 1980 -- 7.8 million square kilometers
September 2012 -- 3.4 million square kilometers



Perovich

Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

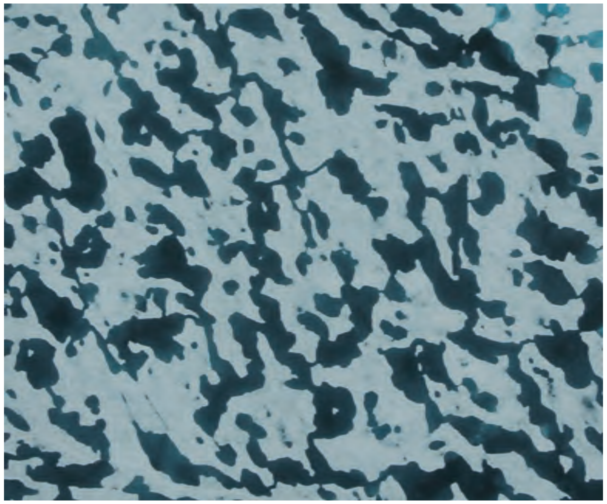
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

account for key processes

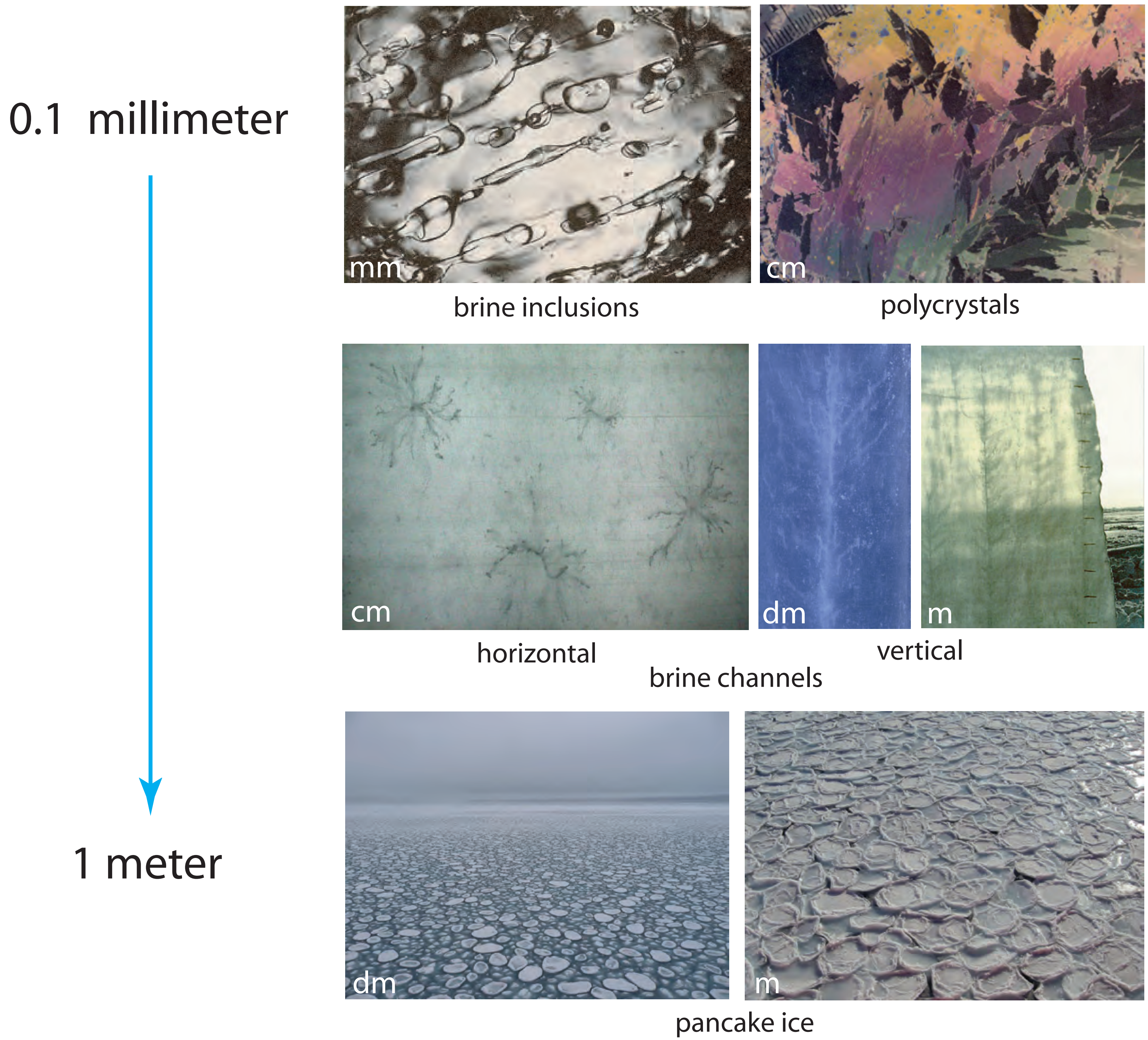
such as melt pond evolution

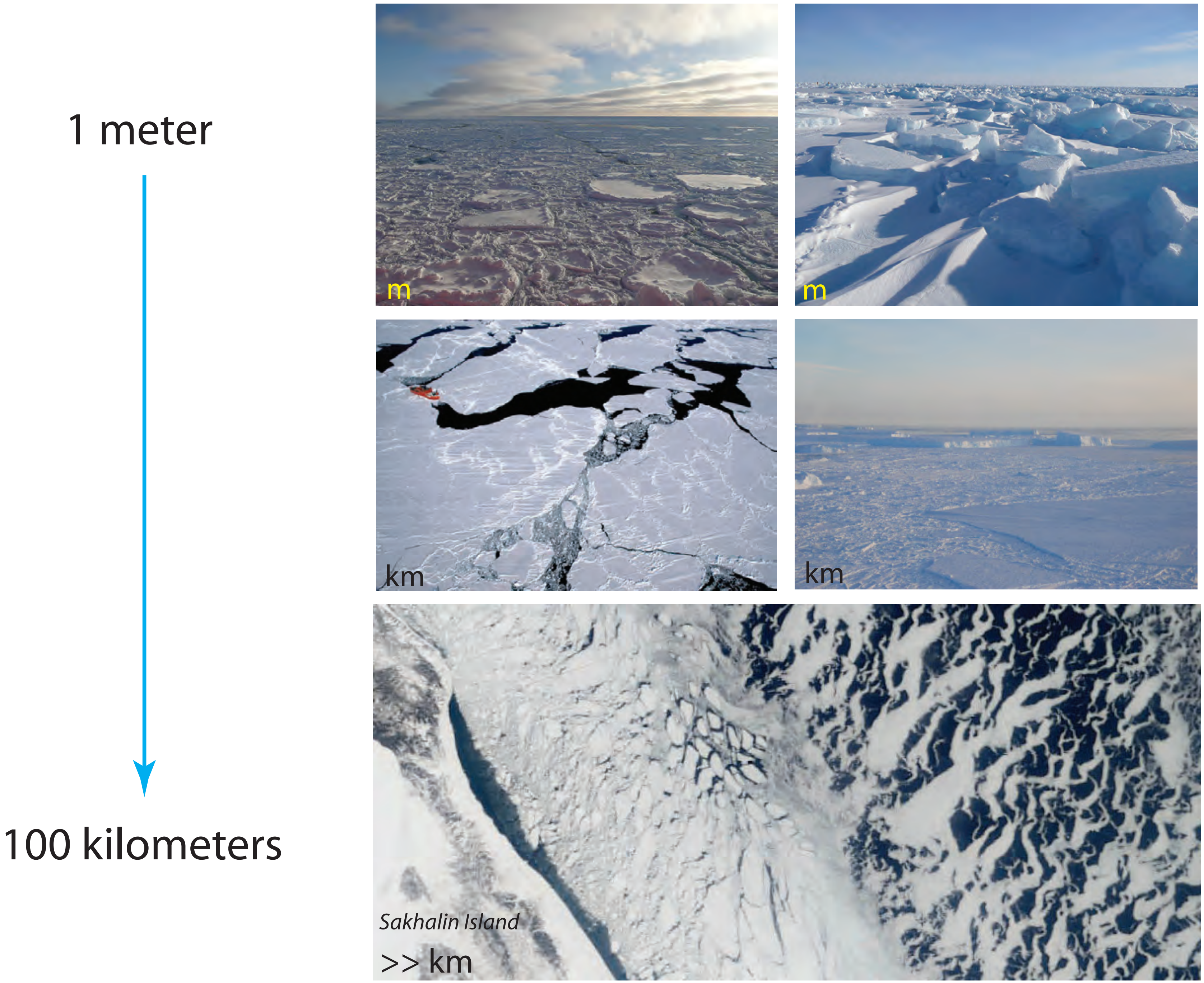


... and other sub-grid scale structures and processes

linkage of scales

sea ice displays ***multiscale*** structure over 10 orders of magnitude




1 meter
100 kilometers
m
m
km
km
Sakhalin Island
>> km

What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

1. ***Fluid flow through sea ice***

   homogenization, diffusion processes, percolation theory

2. ***Electromagnetic monitoring of sea ice***

   complex analysis, spectral measures, random matrix theory

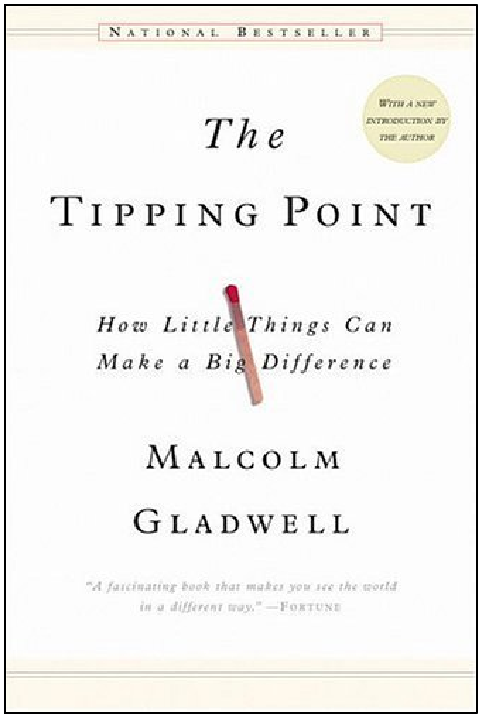
3. ***Fractal geometry of Arctic melt ponds***

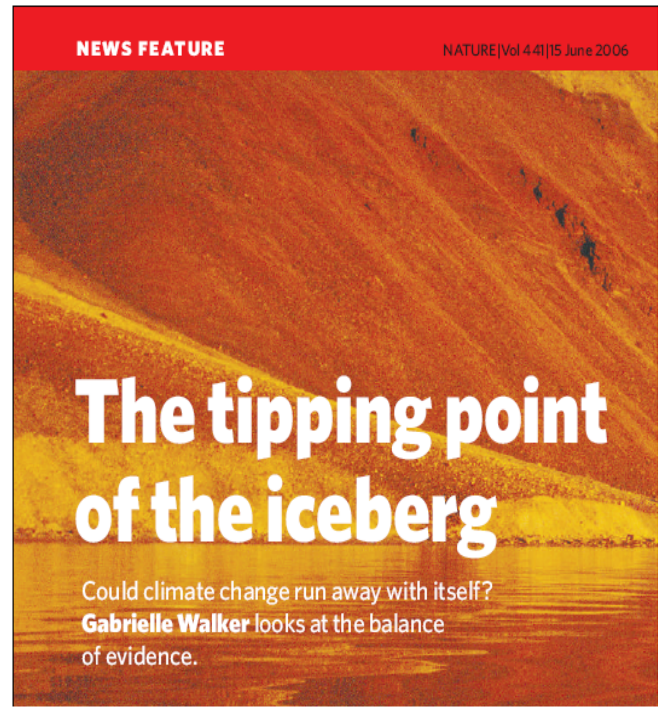
   continuum percolation model, voter model, Ising model

cross - pollination

tipping points in the mainstream

Increasing emphasis in recent years on idea of climate tipping points, with September Arctic sea ice cover receiving much of the attention.





Melting of the Greenland ice sheet
Melting of the West Antarctic ice sheet
Permafrost and tundra loss, leading to the release of methane
Formation of Atlantic deep water near the Arctic ocean **•••**

Lenton, et al., PNAS 2008

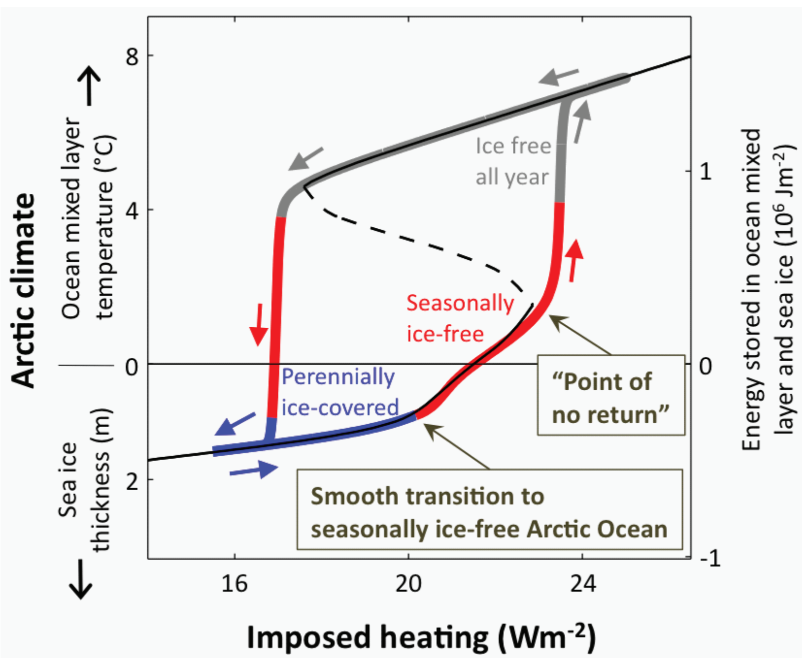
Has Arctic sea ice loss passed through a "tipping point"?

an irreversible downward slide to ice-free Arctic summers (with hysteresis) driven by ice-albedo feedback

low order (toy) models of climate change

Eisenman, Wettlaufer, PNAS 2009:

nonlinear ODE for energy in upper ocean



look for bifurcations in solutions

multiple equilibria: ice-free, ice covered, ...

- tipping point unlikely in loss of summer ice

Abbot, Silber, Pierrehumbert, JGR 2011

bifurcations when include clouds, ice loss

Sudakov, Vakulenko, Golden
Comm. Nonlinear Sci. & Num. Sim., 2014

impact of melt ponds

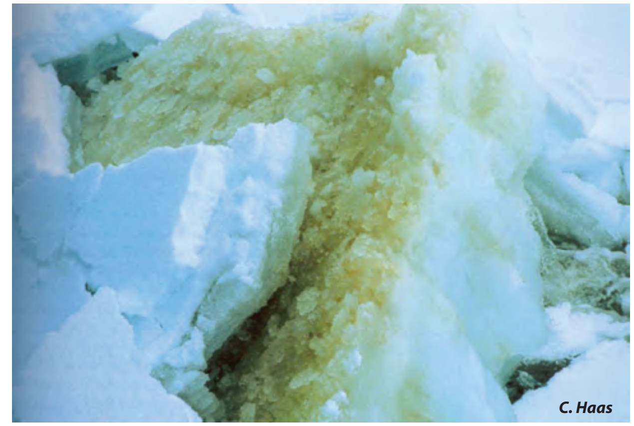
sea ice microphysics

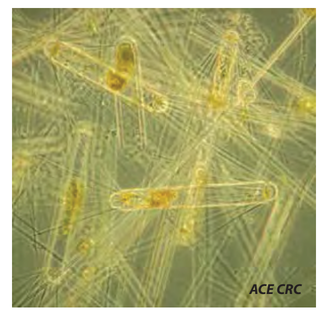
fluid transport

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

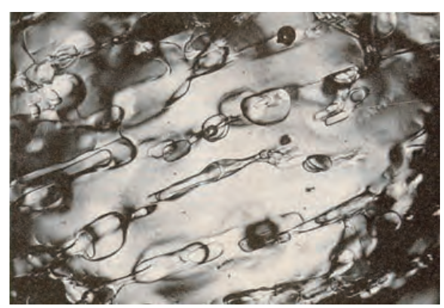
nutrient flux for algal communities





- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat,* CO_2
- *Antarctic surface flooding and snow-ice formation*
- *evolution of salinity profiles*

↕ **linkage of scales**



Darcy's Law for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

viscosity

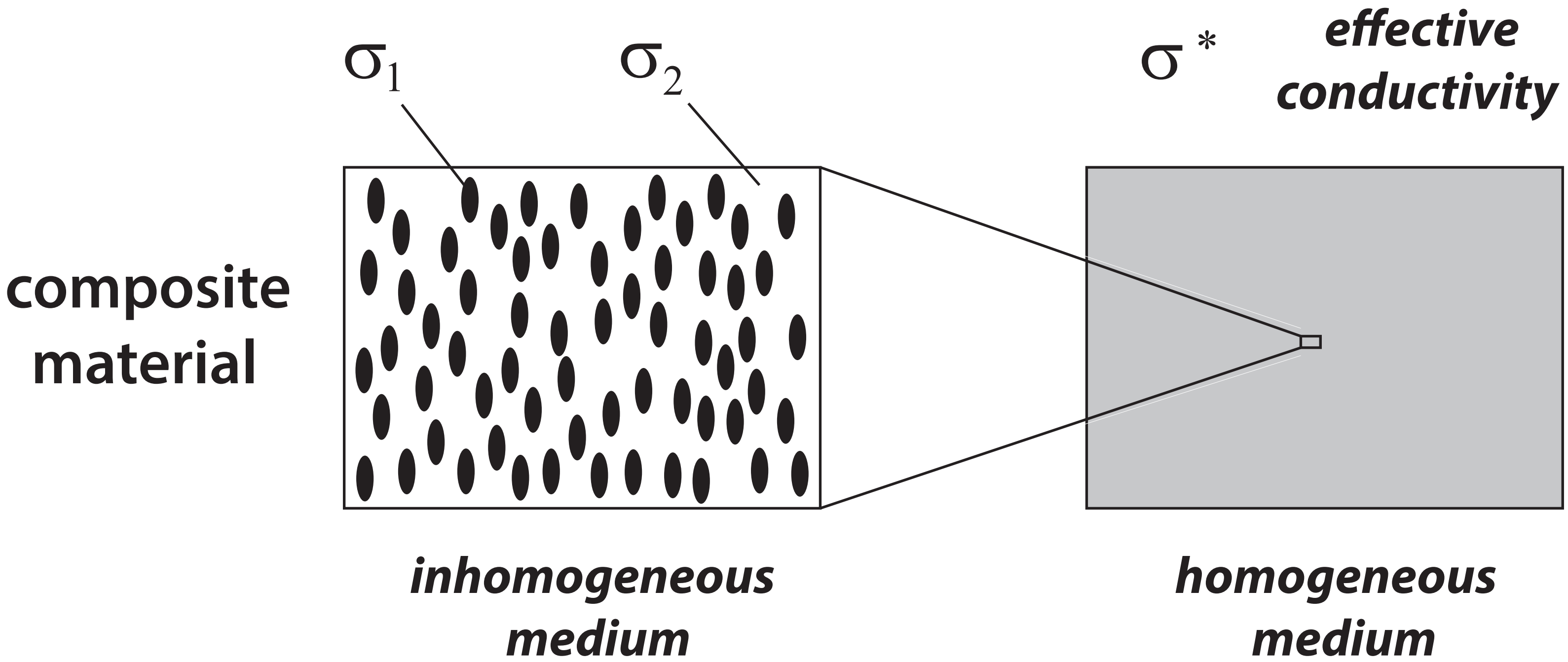
$\mathbf{k}$ = fluid permeability tensor

example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

e.g. transport properties of composites - electrical conductivity, thermal conductivity, etc.

HOMOGENIZATION



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

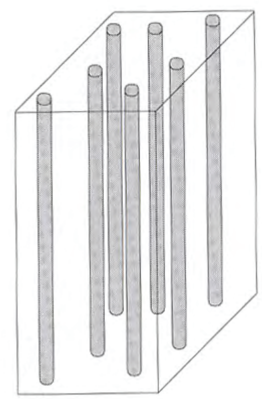
widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

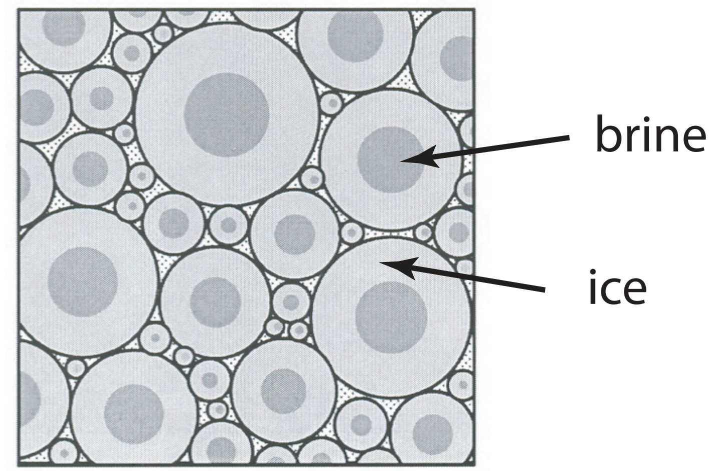
PIPE BOUNDS on vertical fluid permeability k

vertical pipes

with appropriate radii

maximize k

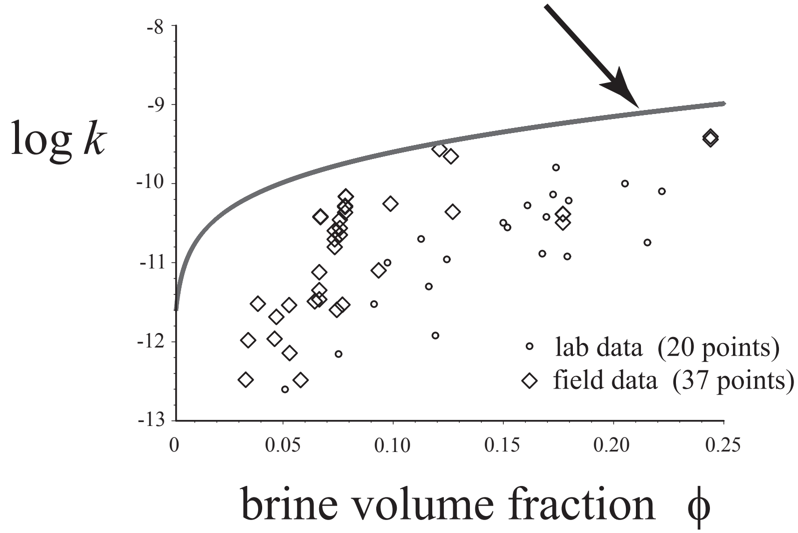




fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)

optimal coated cylinder geometry

lognormal pipe bound



inclusion cross sectional areas A lognormally distributed

$$k \;\leq\; \frac{\phi\langle R^4\rangle}{8\langle R^2\rangle} \;=\; \frac{\phi}{8}\langle R^2\rangle e^{\boldsymbol{\sigma}^2}$$

get bounds through variational analyis of ***trapping constant*** γ for diffusion process in pore space with absorbing BC

$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

for any ergodic porous medium (Torquato 2002, 2004)

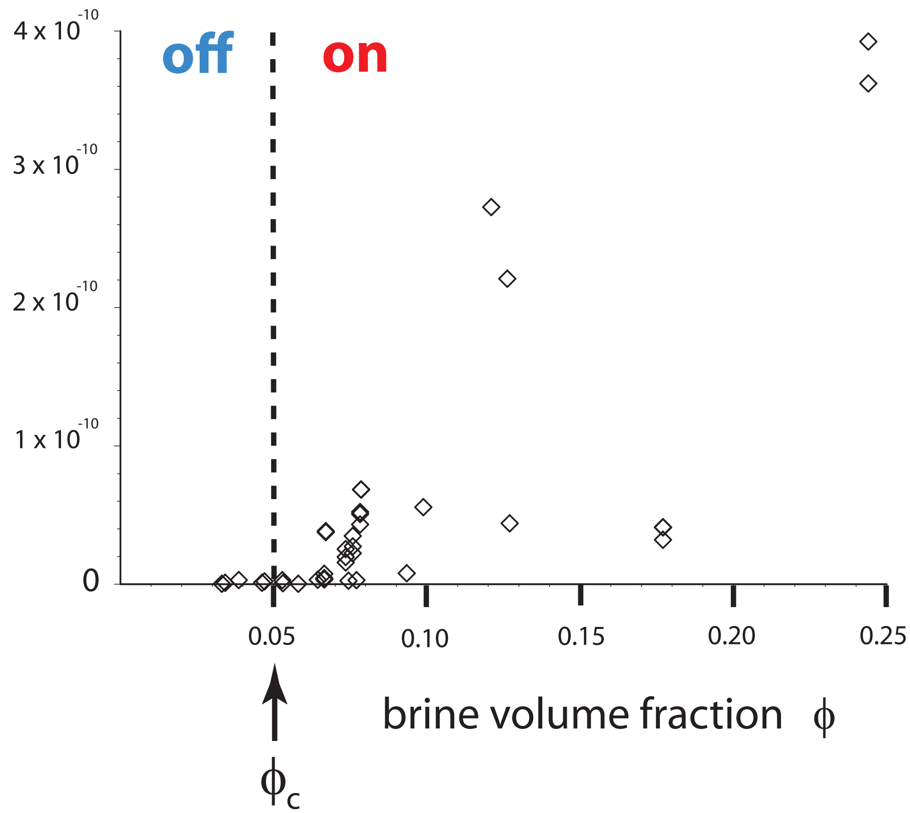
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007
Golden, Heaton, Eicken, Lytle, Mech. Materials 2006

Critical behavior of fluid transport in sea ice

impermeable permeable

Arctic field data

vertical fluid permeability k (m^2)



on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

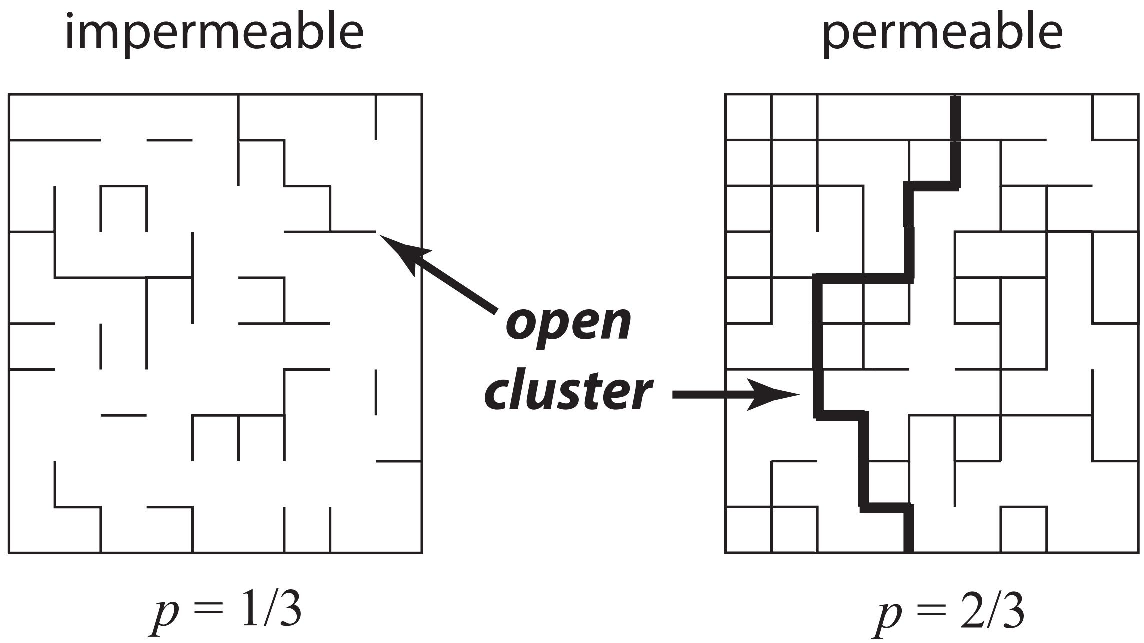
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

Why is the rule of fives true?

percolation theory

probabilistic theory of connectedness



bond → **open** *with probability p*
closed *with probability 1-p*

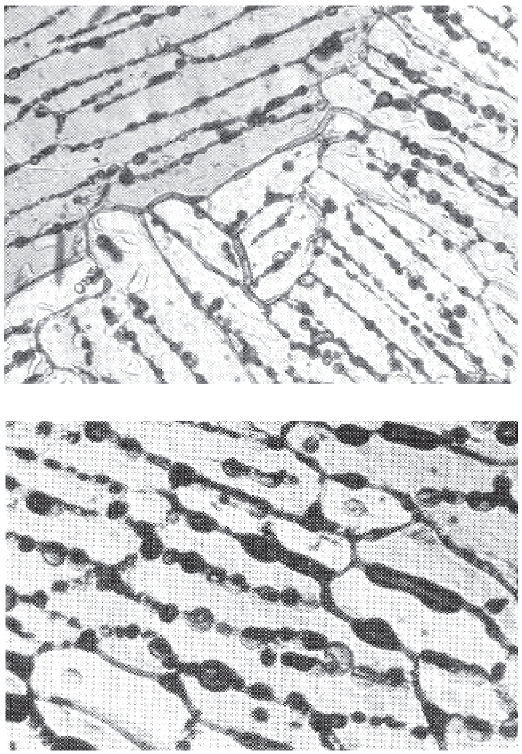
percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

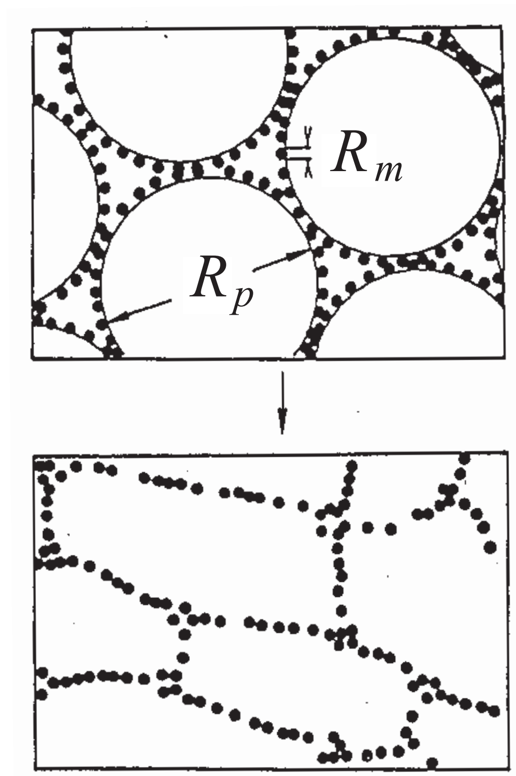
smallest *p* for which there is an infinite open cluster

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

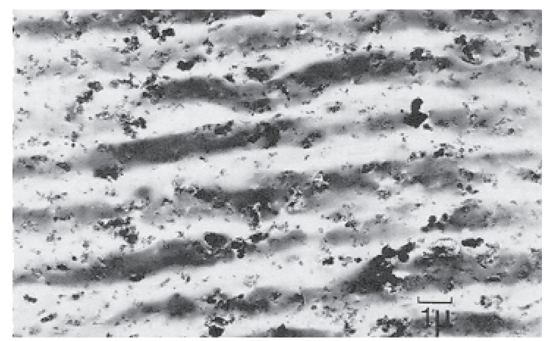
$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science,* 1998



sea ice



compressed powder



radar absorbing composite

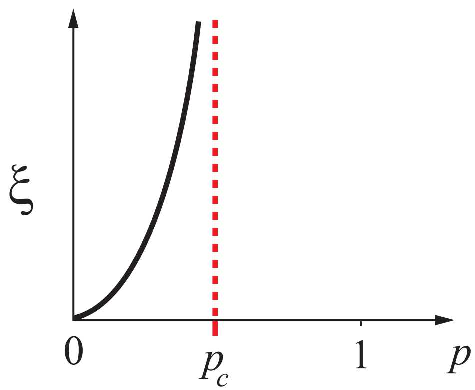
sea ice is radar absorbing

order parameters in percolation theory

geometry

correlation length

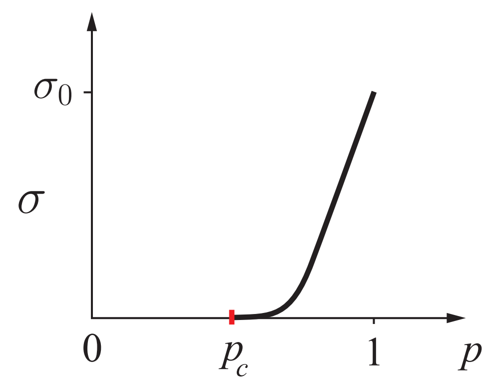
(characteristic scale of connectedness)



$$\xi(p) \sim |p - p_c|^{-\nu}$$

transport

effective conductivity
or fluid permeability



$$\sigma(p) \sim \sigma_0 \, (p - p_c)^t$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \leq t \leq 2$ (for idealized model) **Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992**

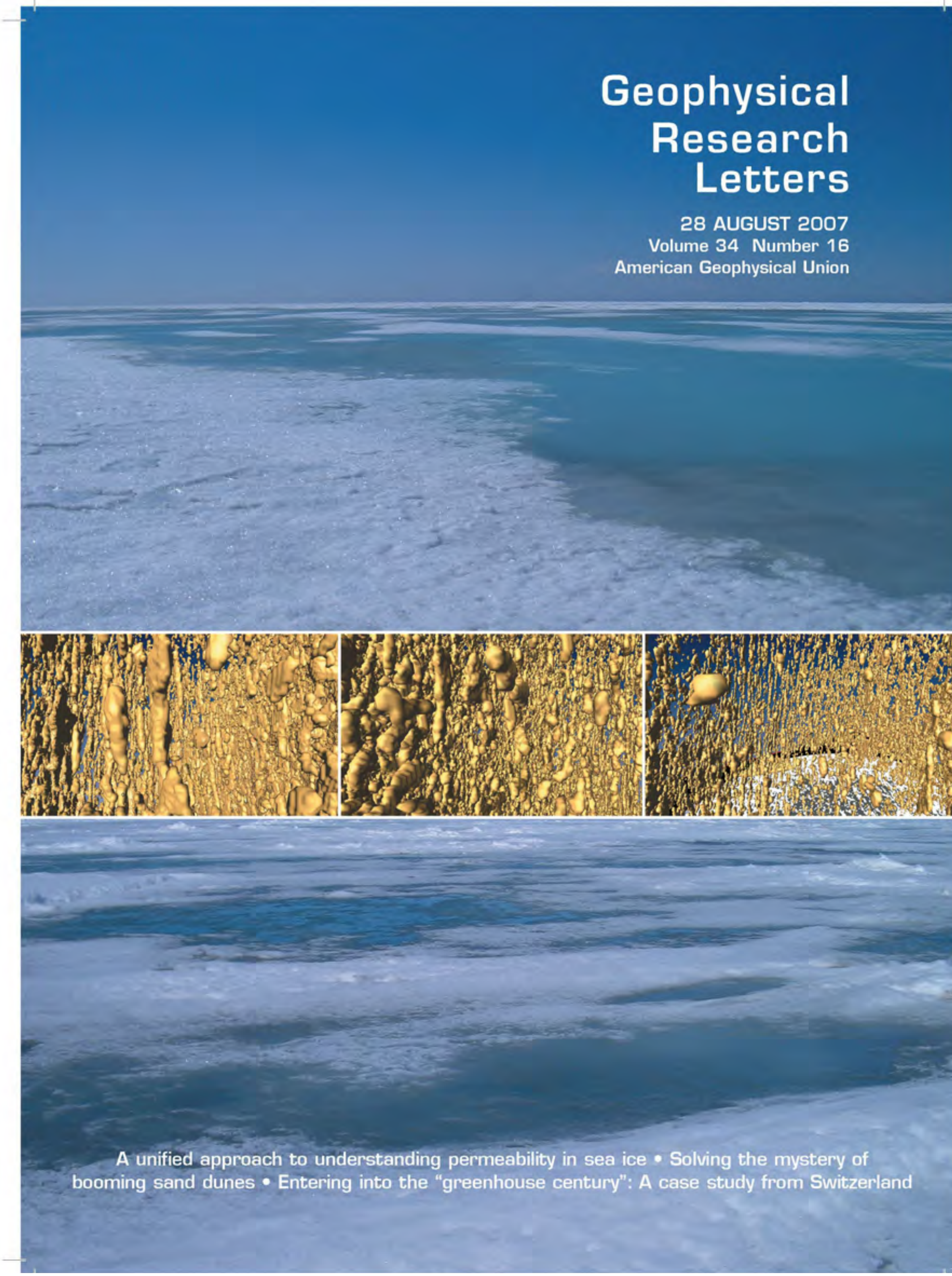
non-universal behavior in continuum

Thermal evolution of permeability and microstructure in sea ice **Golden, Eicken, Heaton, Miner, Pringle, Zhu**

micro-scale

controls

macro-scale processes



rigorous bounds
percolation theory
hierarchical model
network model

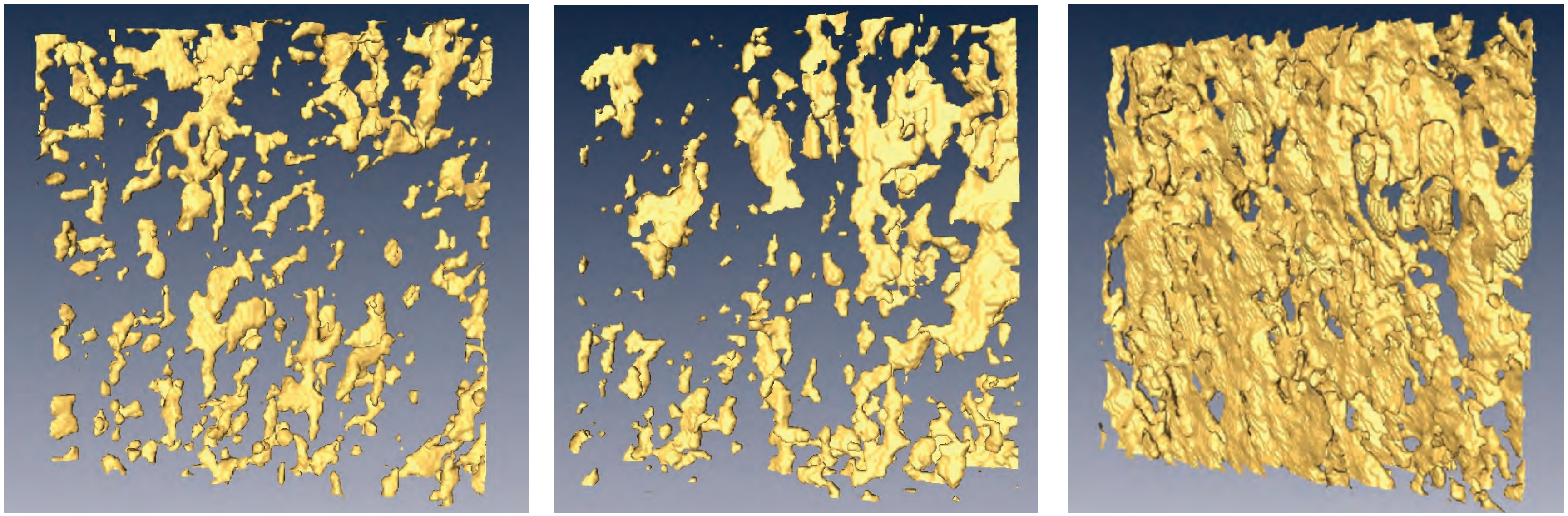
field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

brine connectivity (over cm scale)

8 x 8 x 2 mm



-15 °C, $\phi = 0.033$ -6 °C, $\phi = 0.075$ -3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats → 3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

Pringle, Miner, Eicken, Golden, J. Geophys. Res. 2009

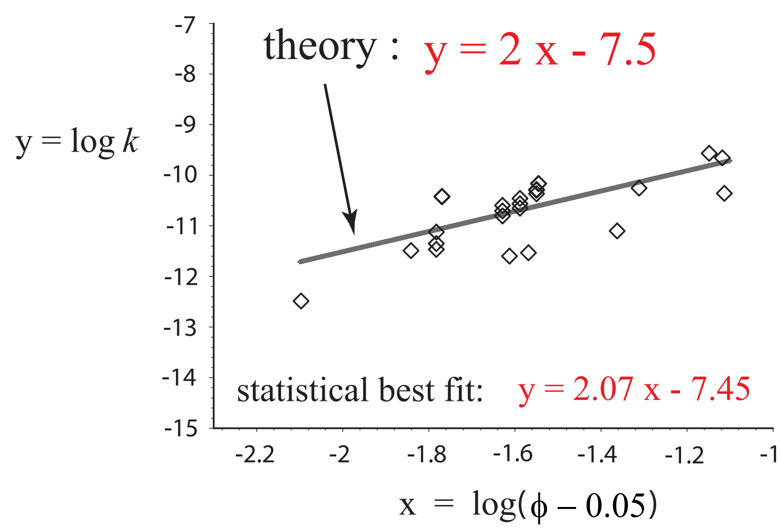
lattice and continuum percolation theories yield:

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8}\ \mathrm{m}^2$$

- exponent is UNIVERSAL lattice value $t \approx 2.0$
- sedimentary rocks like sandstones also exhibit universality
- critical path analysis -- developed for electronic hopping conduction -- yields scaling factor k_0



Remote sensing of sea ice

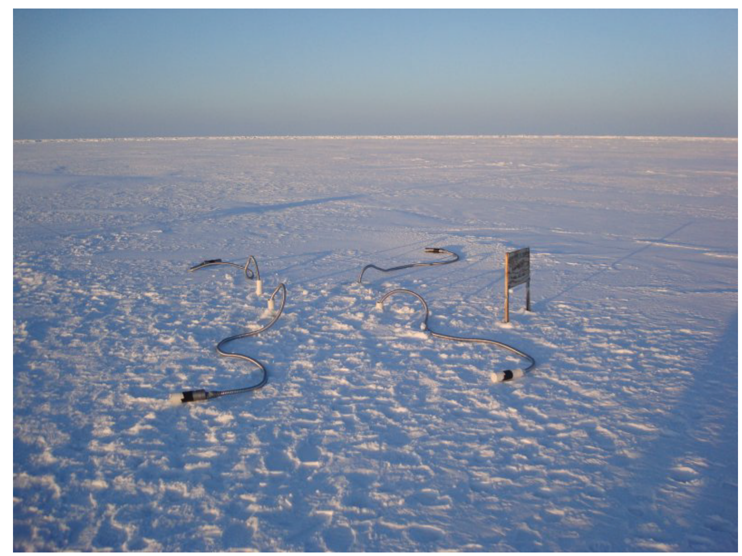
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

$$\varepsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

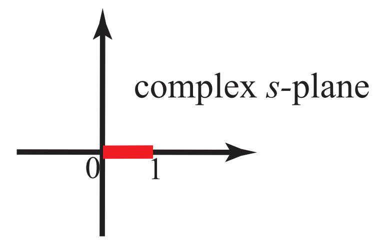
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

composite geometry
(spectral measure μ) → ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
(McPhedran, McKenzie, and Milton, 1982)

ε^* → ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separation of geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

parameters

μ

- spectral measure of self adjoint operator $\chi \Gamma \chi$
- mass = p_1
- higher moments depend on n-point correlations

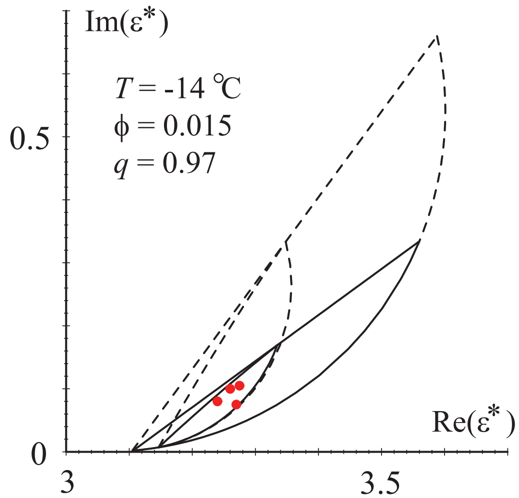
$$\Gamma = -\nabla(-\Delta)^{-1}\nabla \cdot$$

$\chi =$ characteristic function of the brine phase

Golden and Papanicolaou, Comm. Math. Phys. 1983

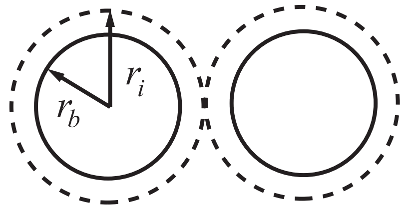
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



$$q = r_b / r_i$$

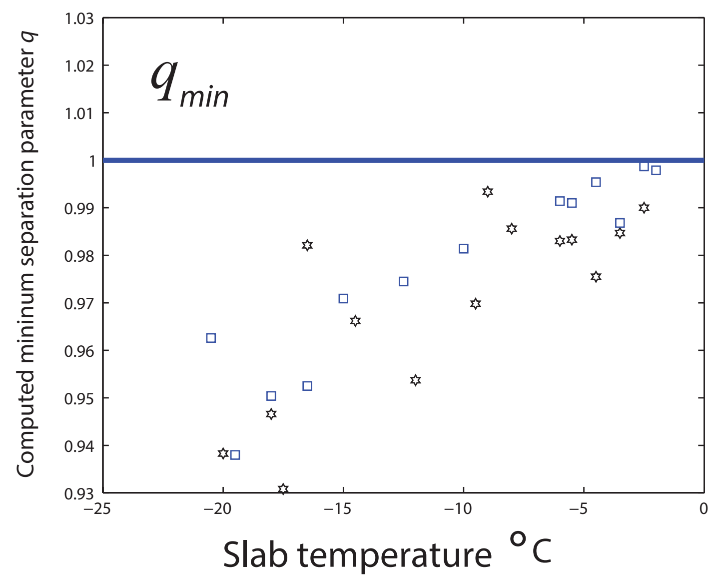
$$0 < q < 1$$

Golden 1995, 1997

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

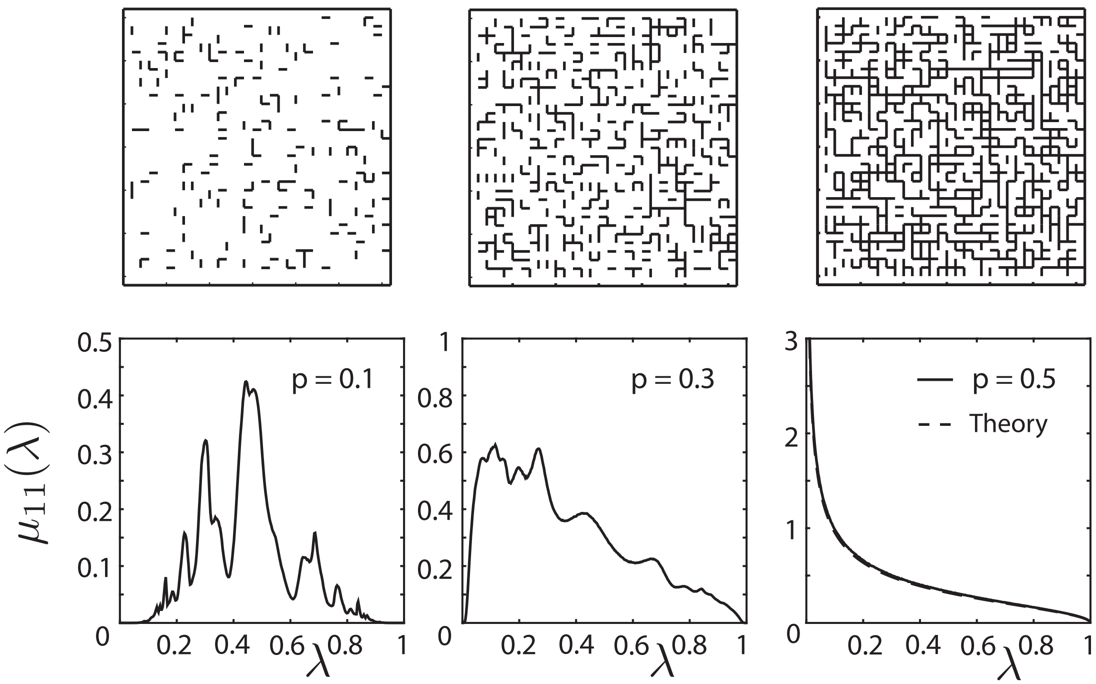
3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

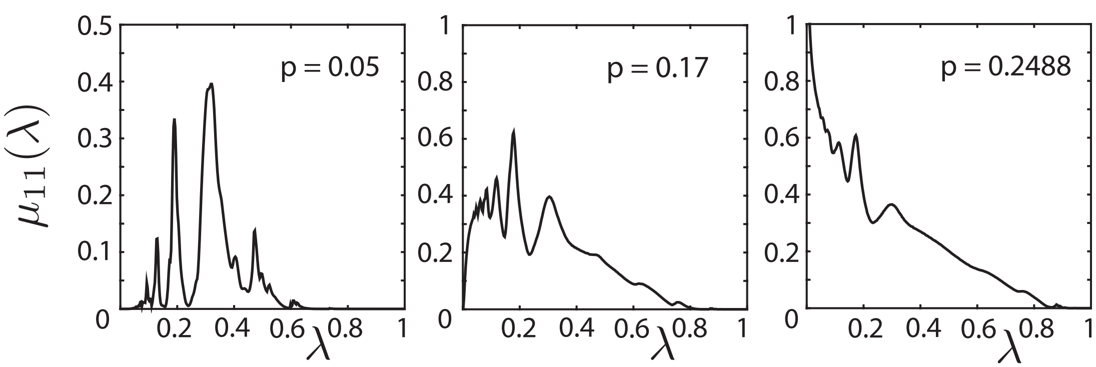
Spectral Measures for Random Resistor Networks

2-D



$p_c = 0.5$

3-D

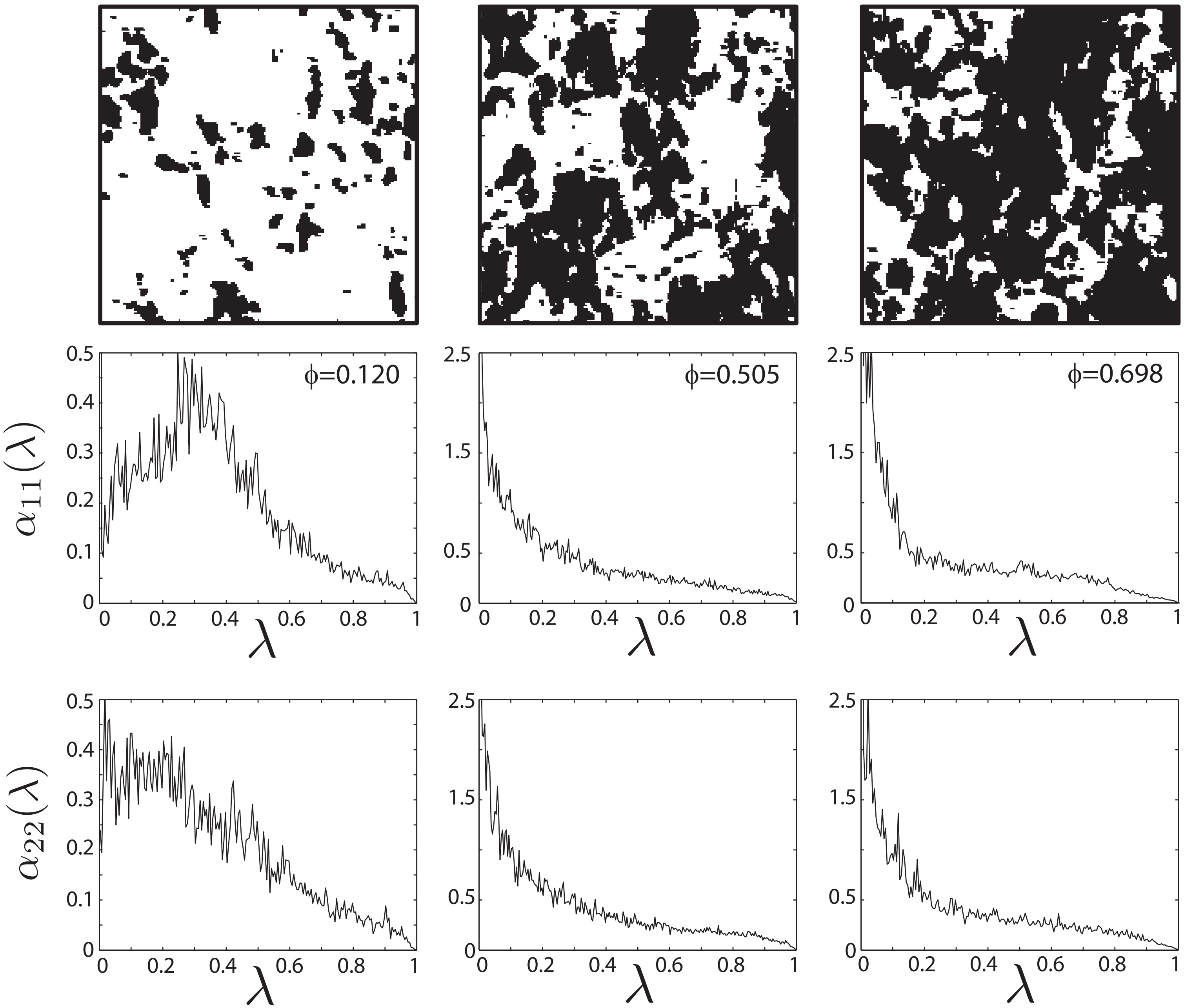


$p_c \approx 0.2488$

spectral gaps collapse at the percolation transitions

Murphy and Golden, J. Math. Phys. (2012)

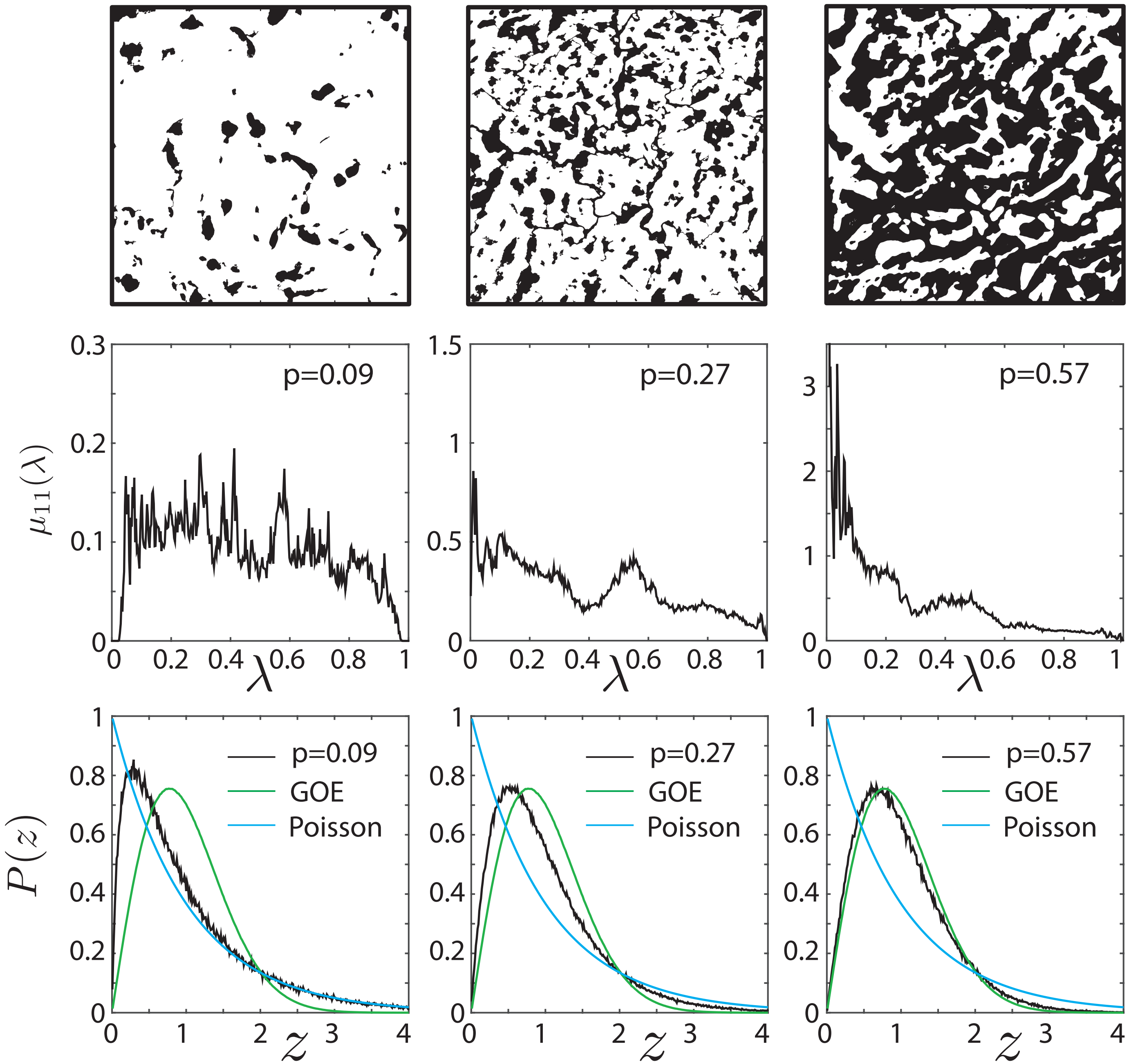
Spectral Measures for Sea Ice Structures: Brine Inclusions



N. B. Murphy, C. Hohenegger, C. S. Sampson, D. K. Perovich, H. Eicken, E. Cherkaev, B. Alali, and K. M. Golden

Spectral computations for Arctic melt ponds

Ben Murphy
Ken Golden
2015



spectral measures

eigenvalue spacing distributions

uncorrelated → **level repulsion**

TRANSITION

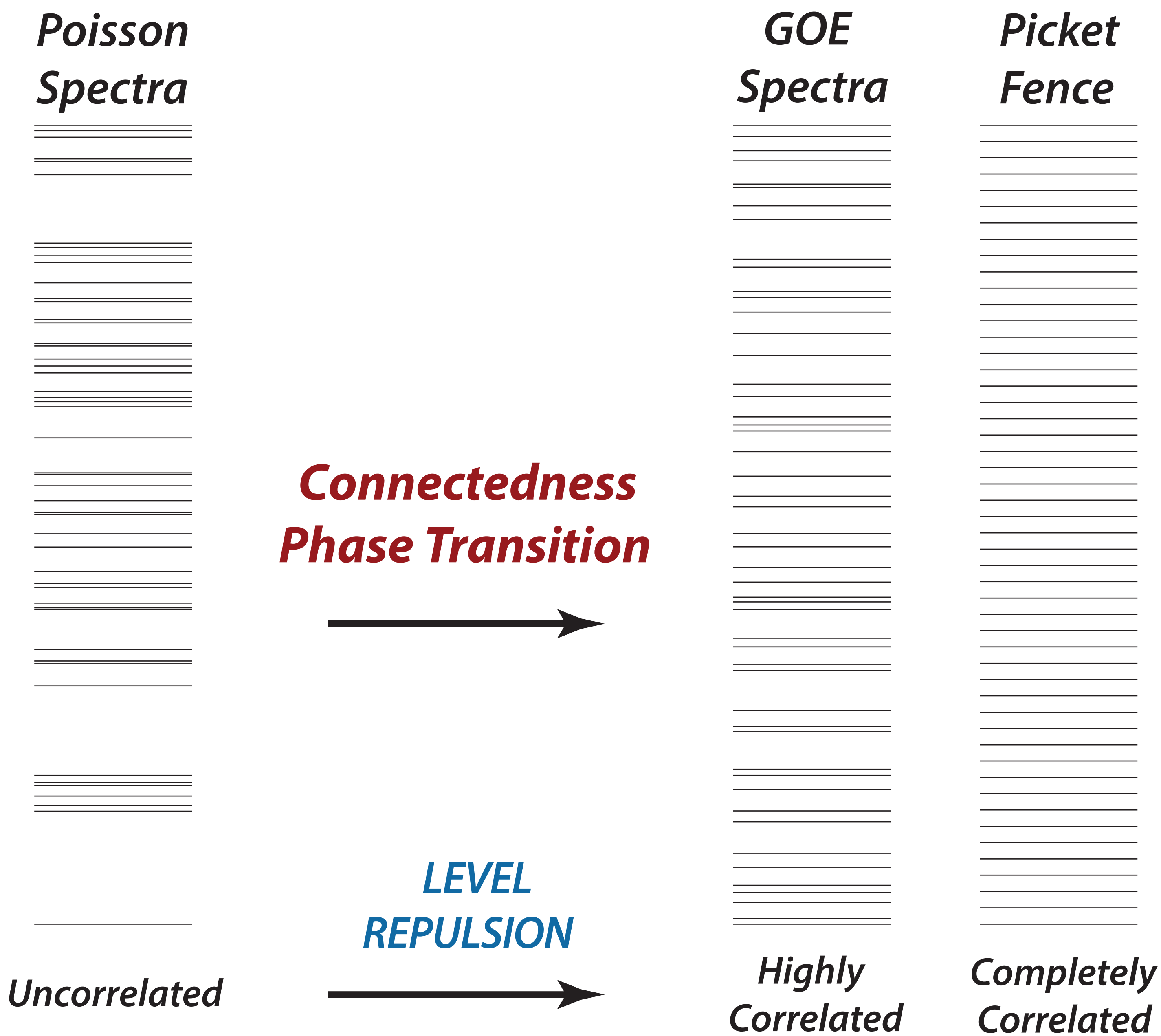
Transition in Eigenvalue Correlations

$$P(z) = \exp(-z)$$

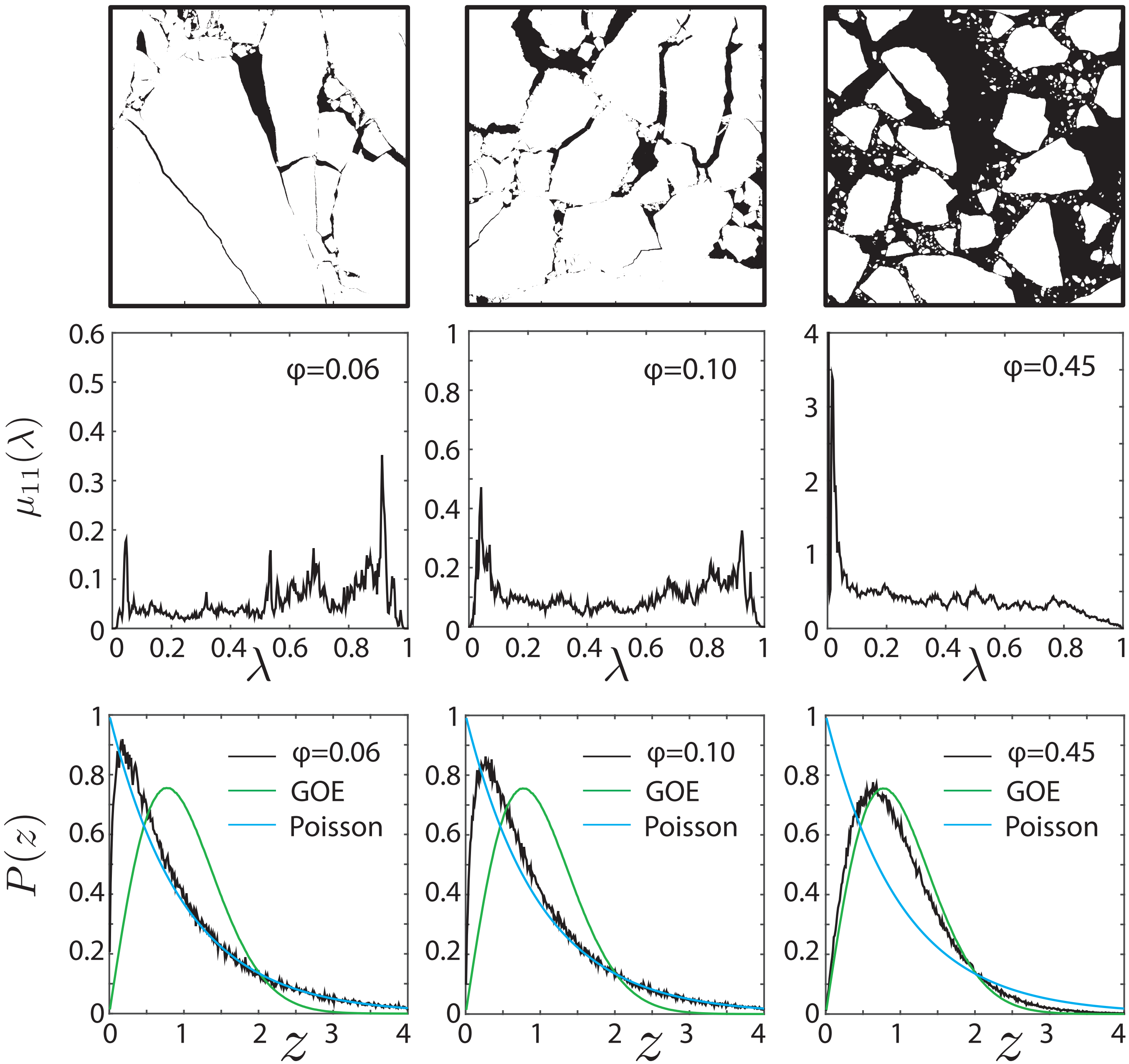
Eigenvalue Spacing Distribution

$$P(z) \approx \frac{\pi z}{2} \exp(-\pi z^2/4)$$ **Wigner surmise**

Eigenvalue Spacing Distribution



Spectral computations for Arctic sea ice pack



spectral measures

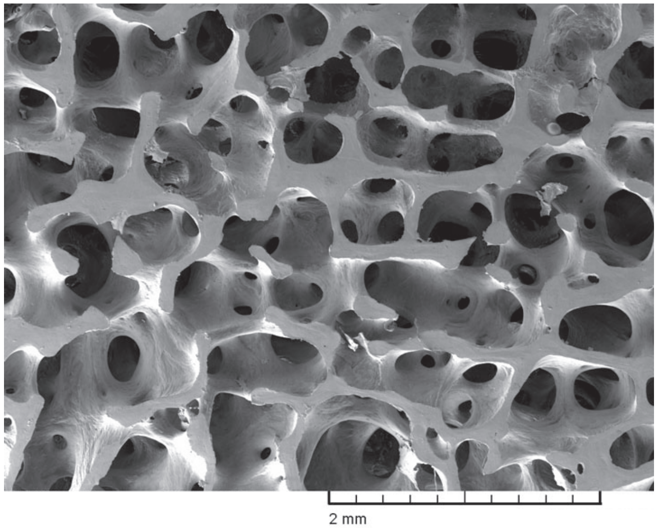
eigenvalue spacing distributions

TRANSITION

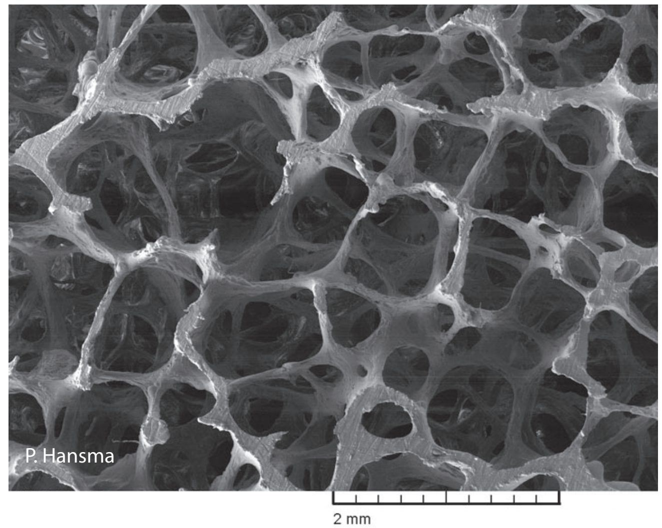
spectral characterization of porous microstructures in bone

Golden, Murphy, Cherkaev, J. Biomechanics 2011

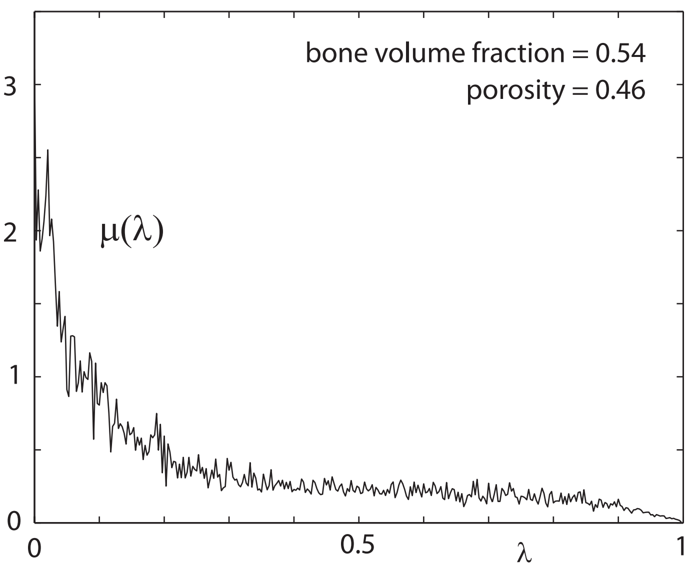
(a) young healthy trabecular bone



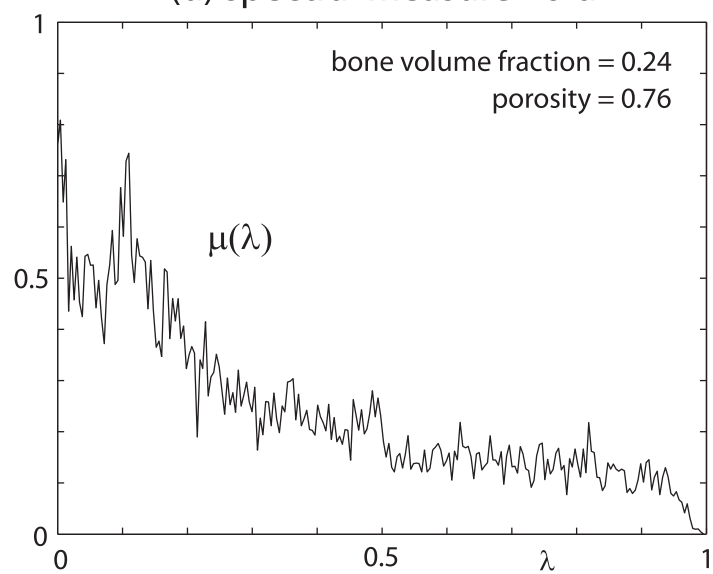
(b) old osteoporotic trabecular bone



(c) spectral measure - young



(d) spectral measure - old



+

reconstruction of spectral measures from complex permittivity data

using regularized inversion scheme

↓

EM monitoring of osteoporosis

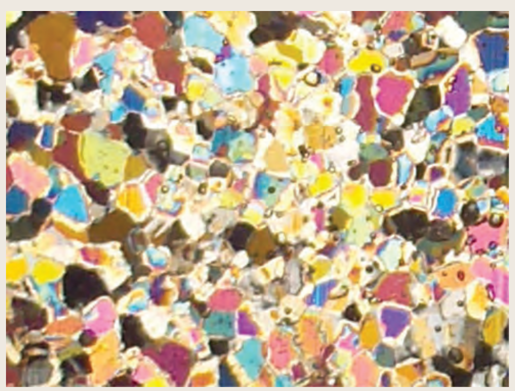
loss of bone connectivity

the math doesn't care if it's sea ice or bone!

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A   8 Feb 2015

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PROCEEDINGS A

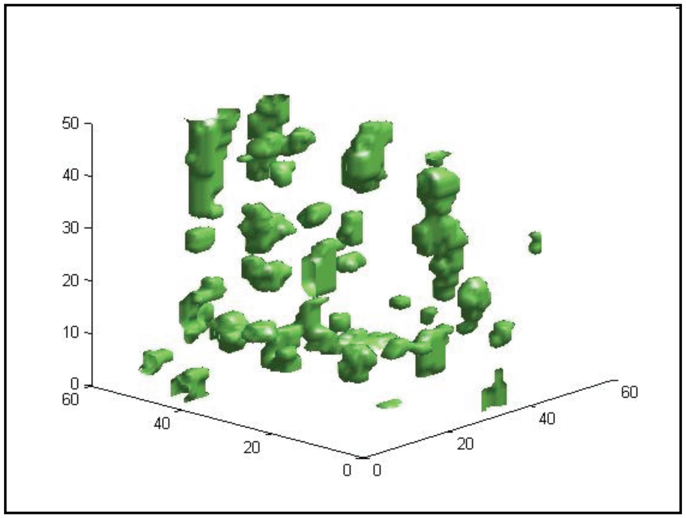
350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

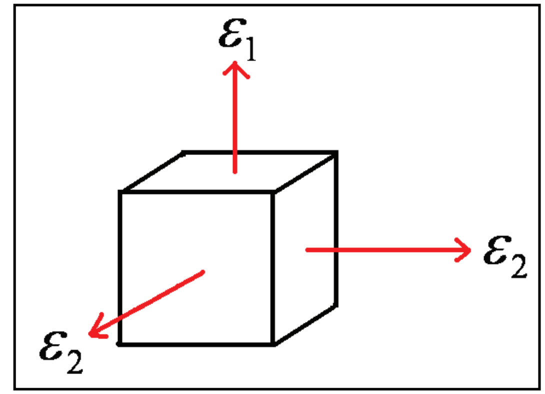
A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

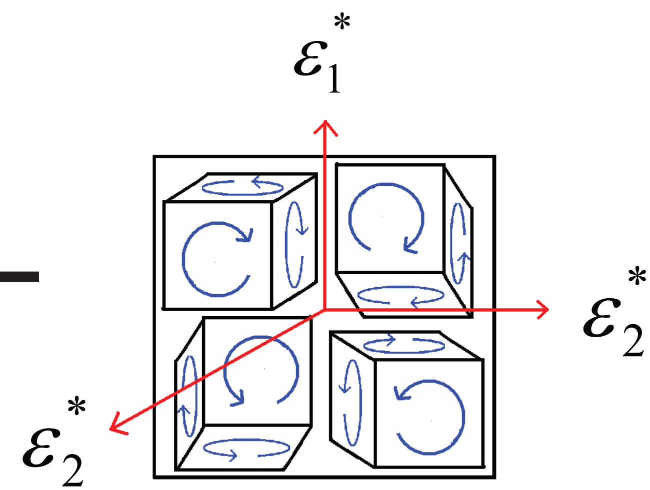
two scale homogenization for polycrystalline sea ice



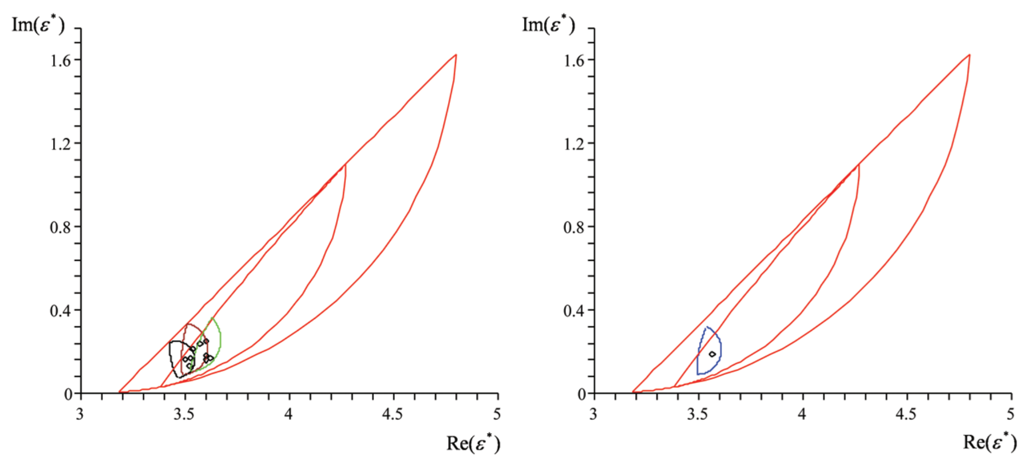
numerical homogenization
for single crystal



analytic continuation
for polycrystals



bounds

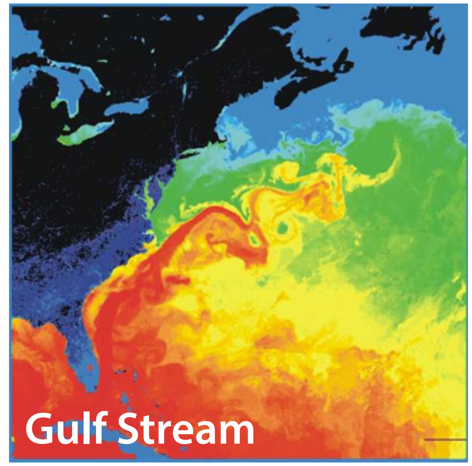


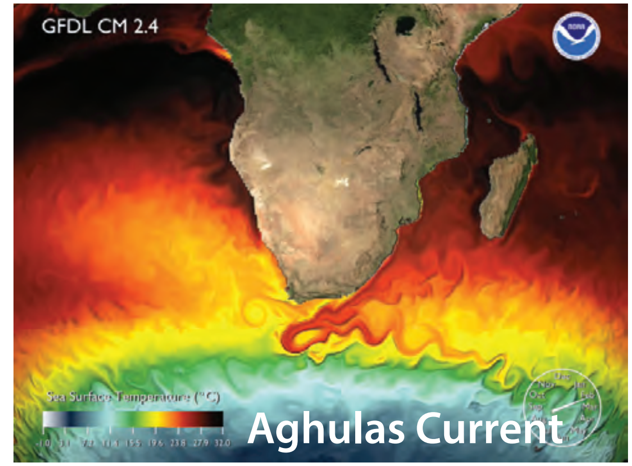
Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

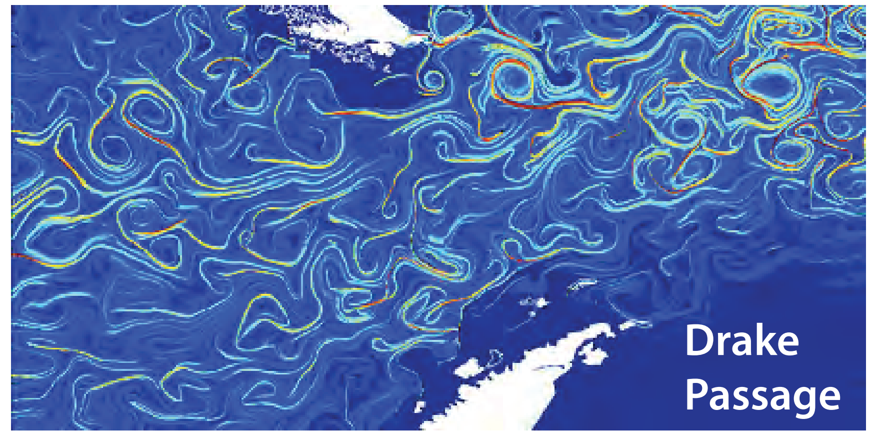
advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean







advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

↓ ***homogenize***

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity*** *analytic function of Péclet number*

Stieltjes integral for κ^* ***with spectral measure***

Avellaneda and Majda, PRL 89, CMP 91

Stieltjes integral for κ^* with spectral measure

composites

Golden and Papanicolaou, CMP 1983

$$\frac{\epsilon^*}{\epsilon_2} = 1 - \int_0^1 \frac{d\mu(\lambda)}{s-\lambda}$$

$$s = \frac{1}{1 - \epsilon_1/\epsilon_2}$$

advection diffusion

Avellaneda and Majda, PRL 89, CMP 91

$$\frac{\kappa^*}{\kappa_0} = 1 - \int_0^\infty \frac{d\rho(z)}{t-z}$$

$t = -1/\xi^2$, ξ = Péclet number

- computations of spectral measures and effective diffusivity for model flows

iΓHΓ $\quad \vec{u} = \kappa_0 \, \xi \, \vec{\nabla} \cdot \mathbf{H}$

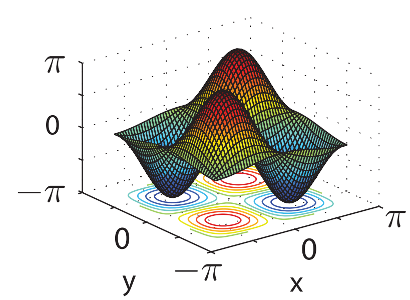
$\mathbf{H}$ antisymmetric vector potential

Murphy, Cherkaev, Zhu, Xin, Golden 2015

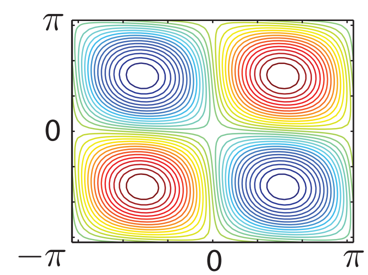
- rigorous bounds and computations on convection enhanced thermal conductivity of sea ice

Wang, Liu, Zhu, Golden 2015

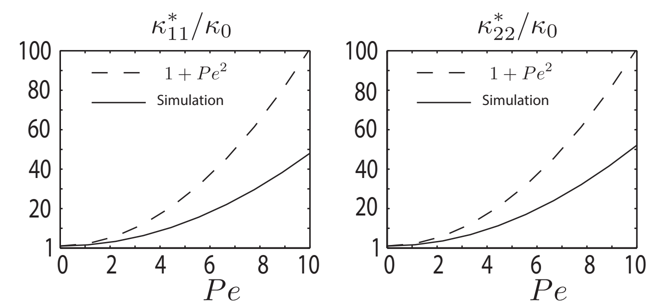
stream function



streamlines



effective diffusivities



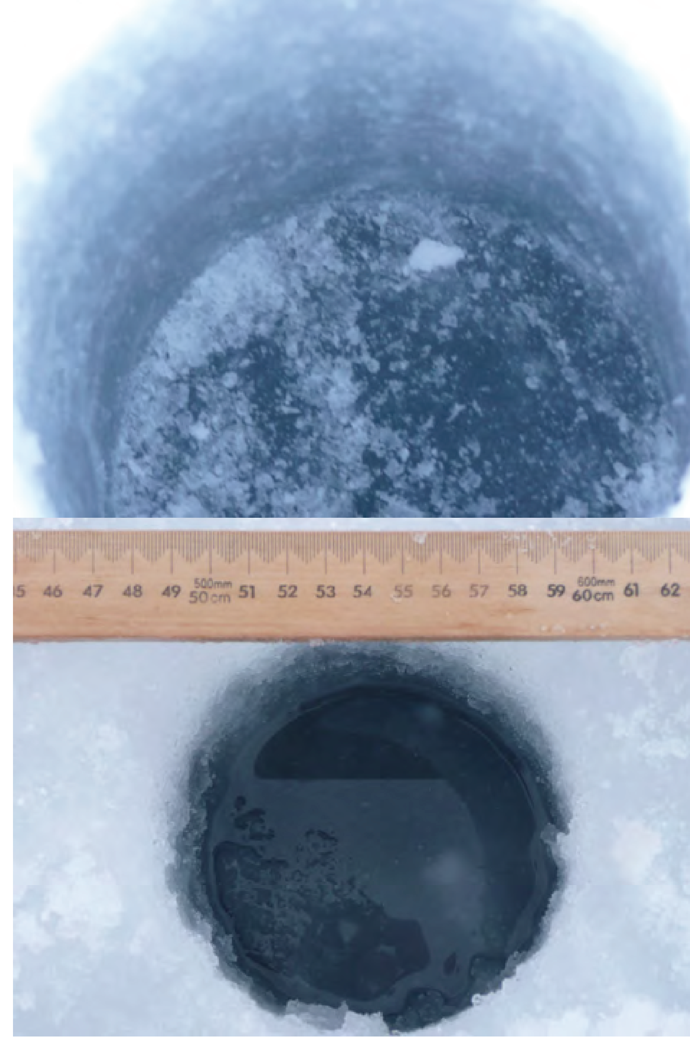
Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

photo by Jan Lieser

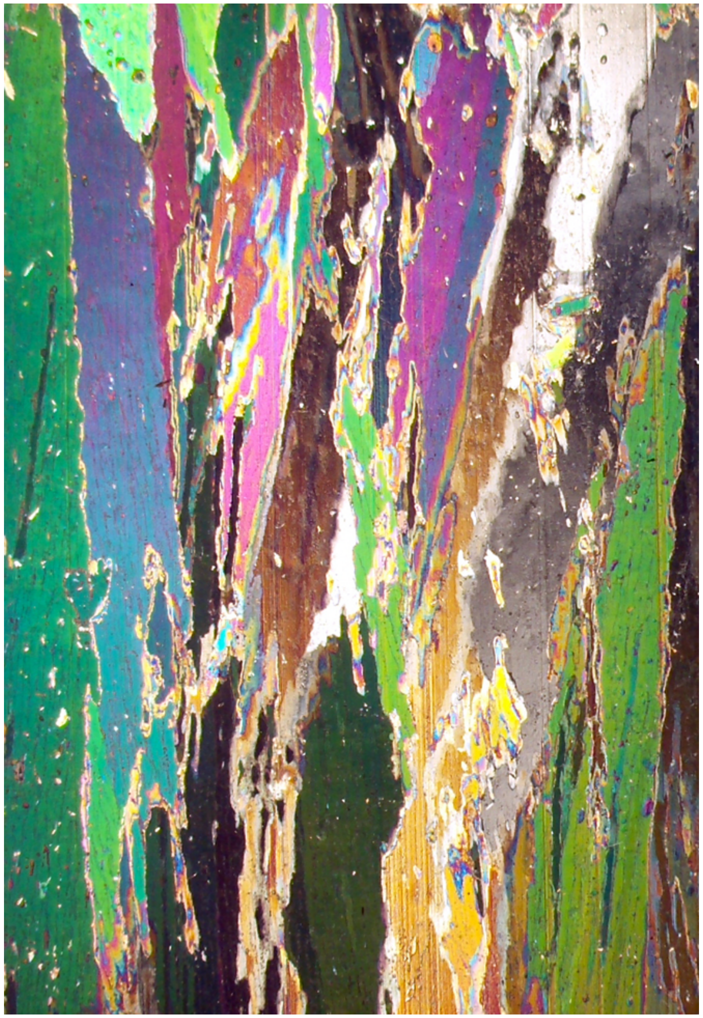


measuring fluid permeability of Antarctic sea ice

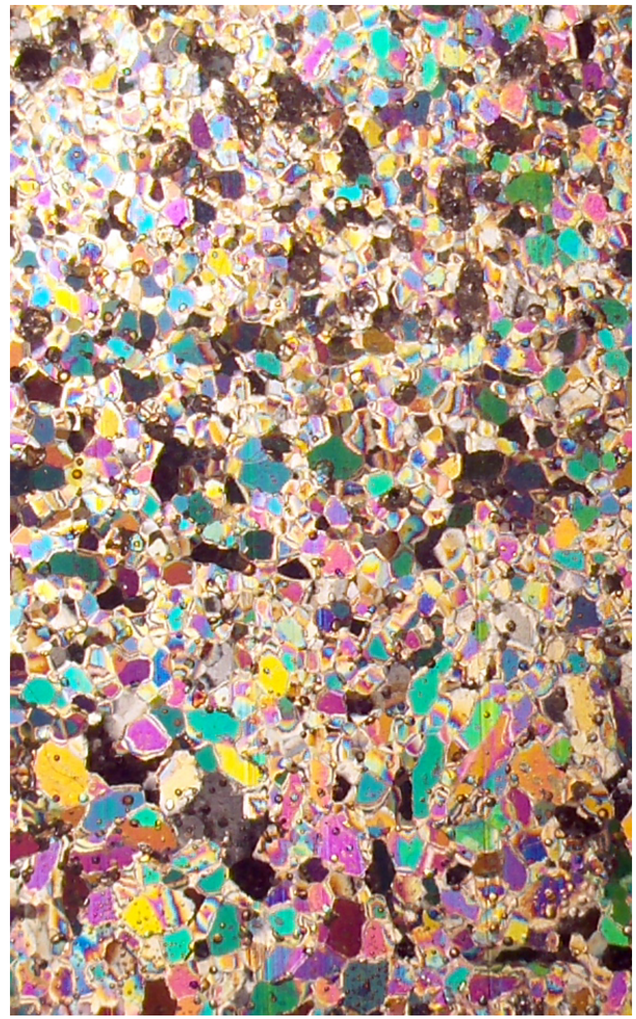
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar



granular

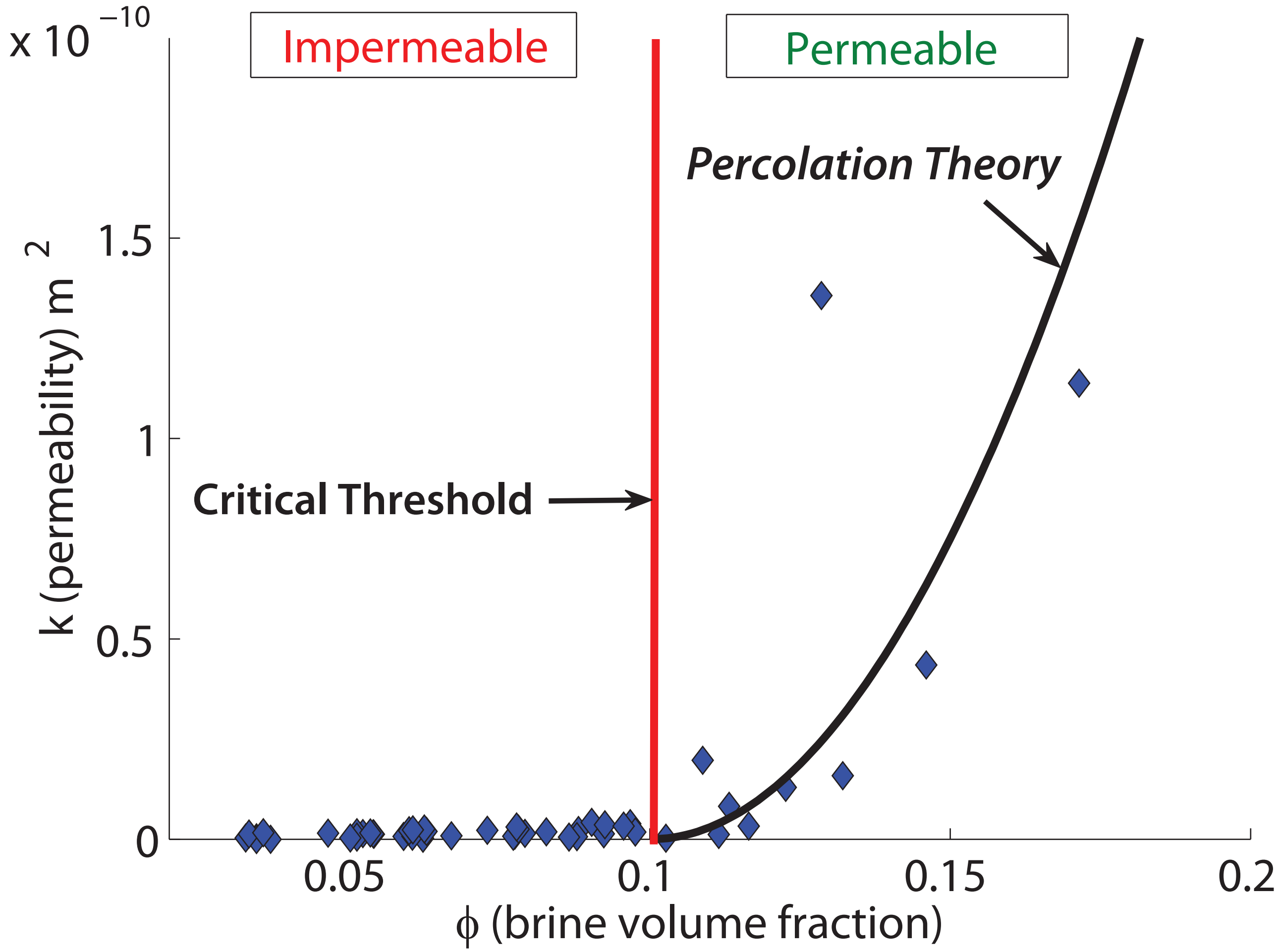


5%

10%

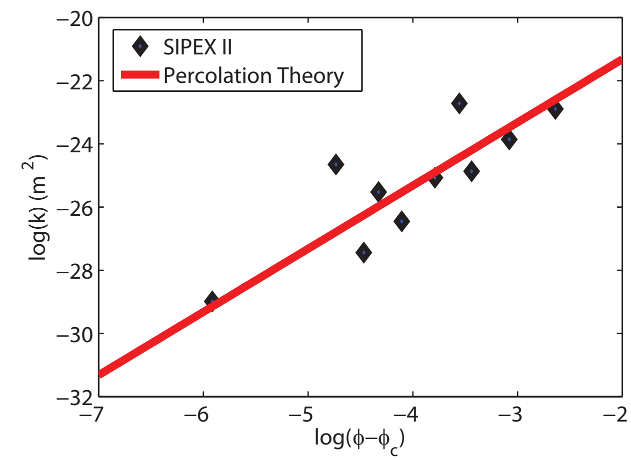
Golden, Sampson, Gully, Lubbers, Tison 2015

SIPEX II vertical permeability data



same universal critical exponent as lattice models

data above threshold

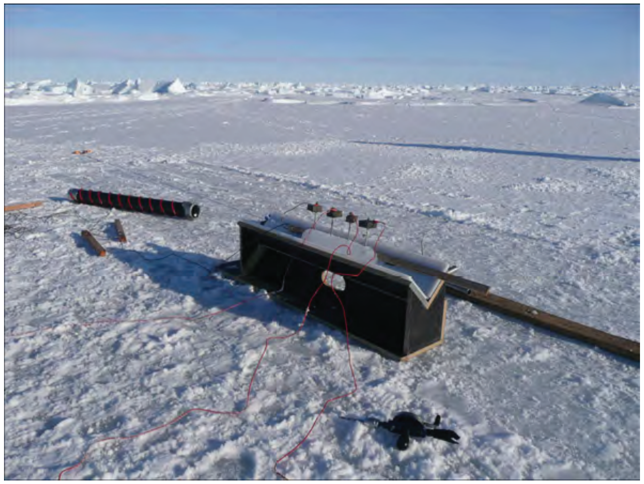


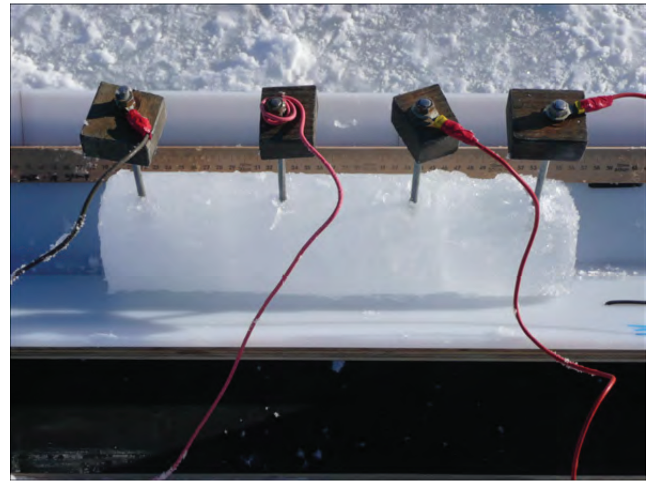
higher threshold in granular ice predicted with percolation theory by Golden, et al. (Science, 1998)

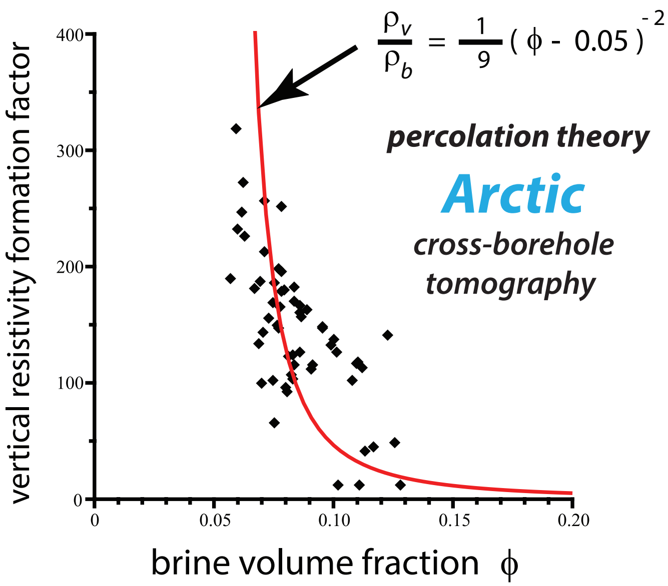
not confirmed experimentally until SIPEX I (2007) and SIPEX II (2012)

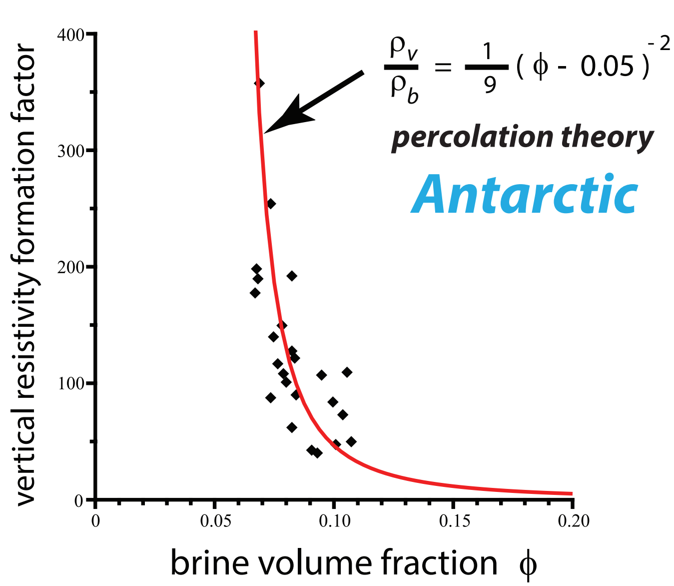
critical behavior of electrical transport in sea ice

electrical signature of the on-off switch for fluid flow







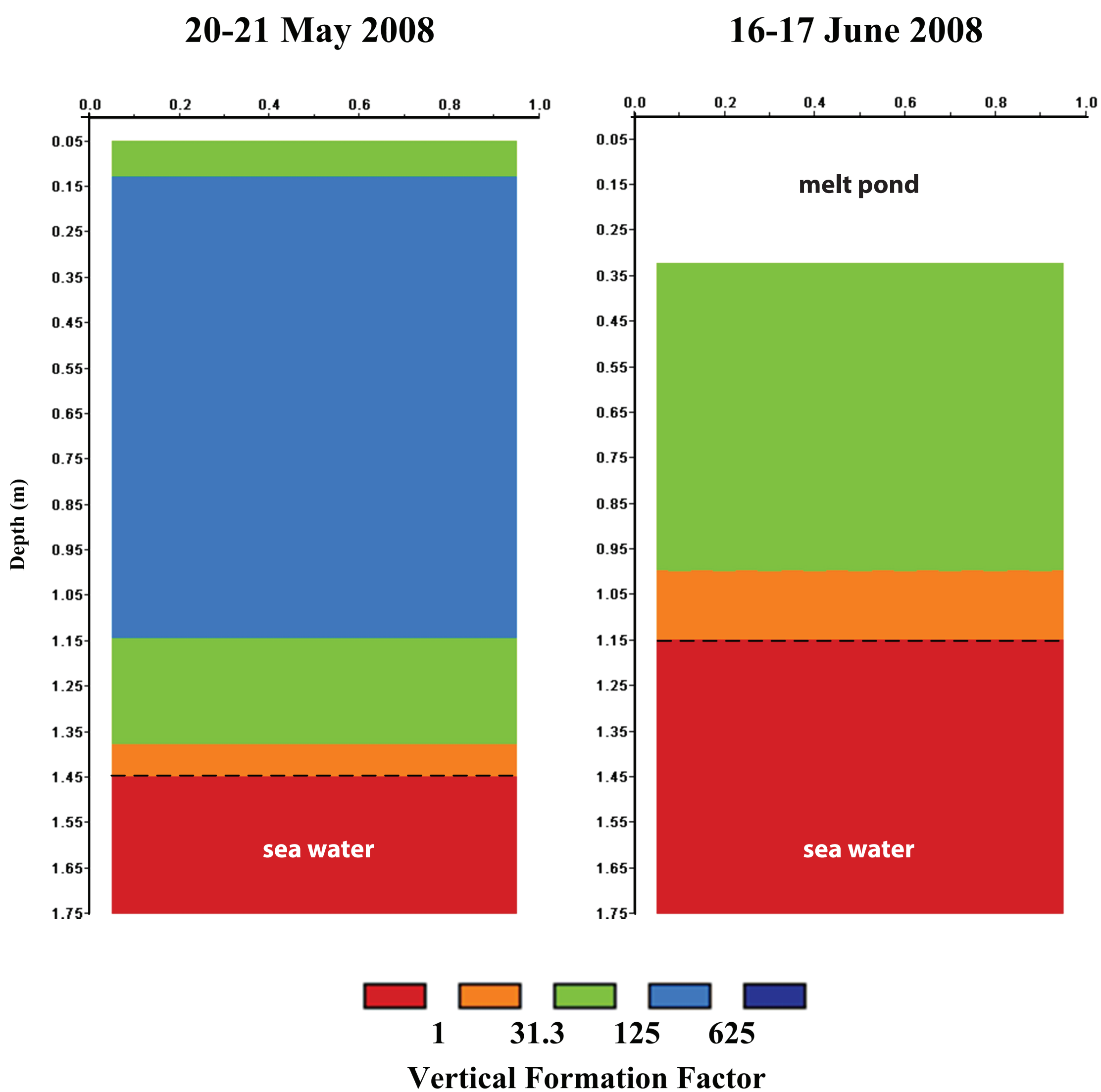


cross-borehole tomography - electrical classification of sea ice layers

Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2015

Cross-borehole tomographic reconstructions of sea ice resistivity

before and after melt pond formation



impermeable - ponds can grow

impermeable for CO2 exchange & build-up of nutrients and biomass; supports surface ponding

highly permeable - CO2 pumping, build-up of nutrients and biomass

Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, and Worby 2015

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

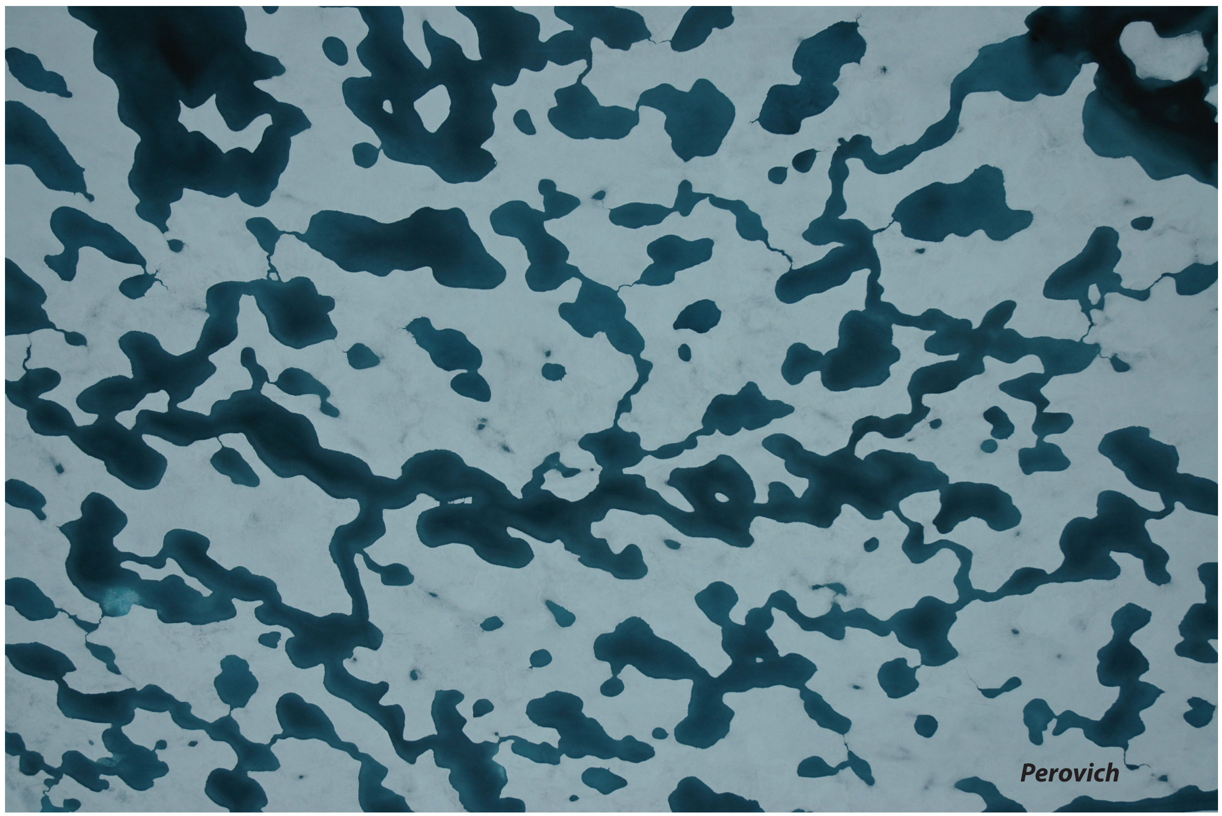
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

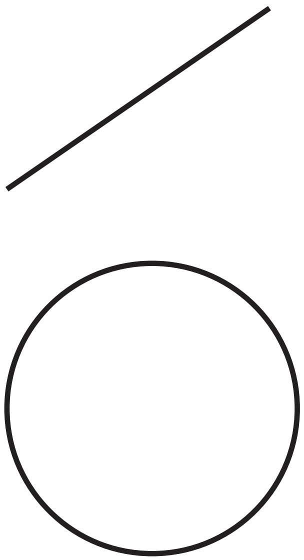
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

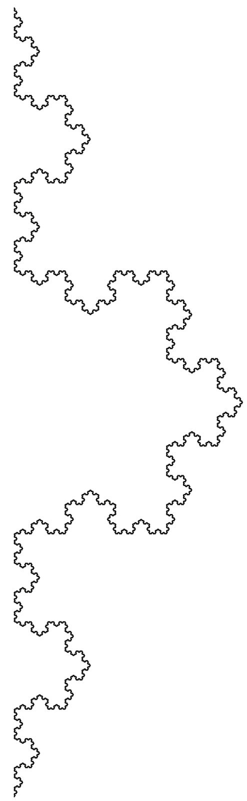
fractal curves in the plane

they wiggle so much that their dimension is >1



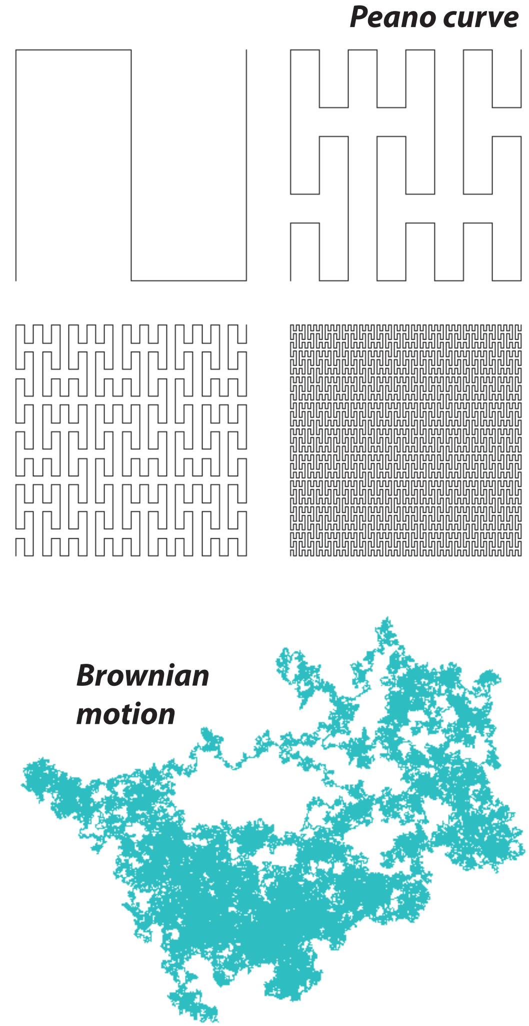
simple curves

D = 1



Koch snowflake

D = 1.26



space filling curves

D = 2

clouds exhibit fractal behavior from 1 to 1000 km

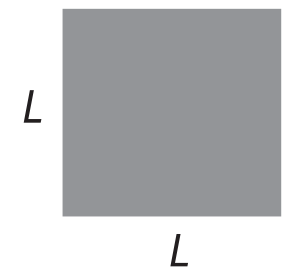
use ***perimeter-area*** data to find that cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982

simple shapes

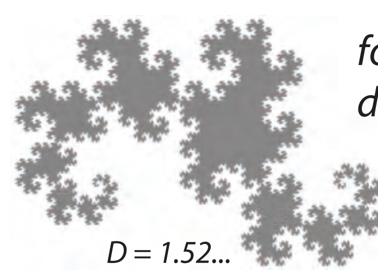
$$P \sim \sqrt{A}$$



$A = L^2$

$P = 4L = 4\sqrt{A}$

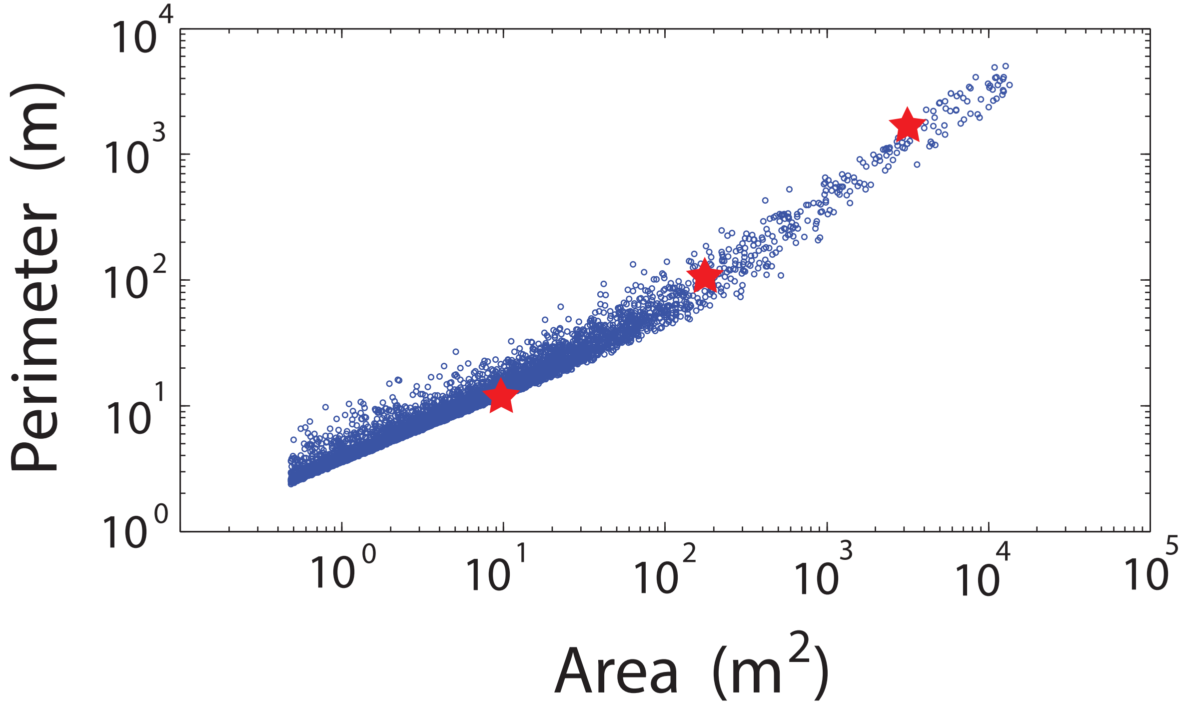
$$P \sim \sqrt{A}^{D}$$



for fractals with dimension D

Transition in the fractal geometry of Arctic melt ponds *The Cryosphere, 2012*

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

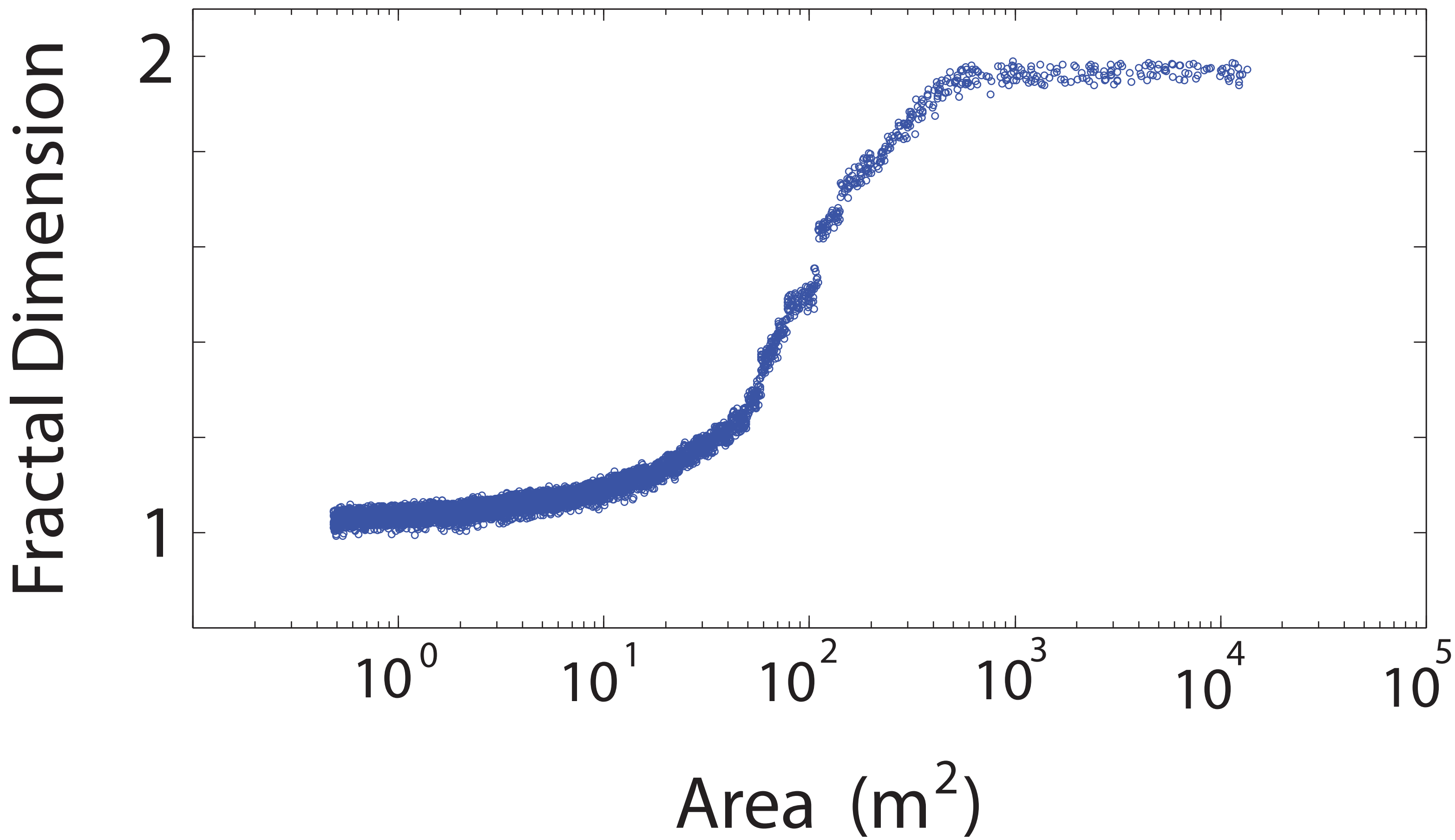


data from 5269 Arctic melt ponds

simple pond ***transitional pond*** ***complex pond***

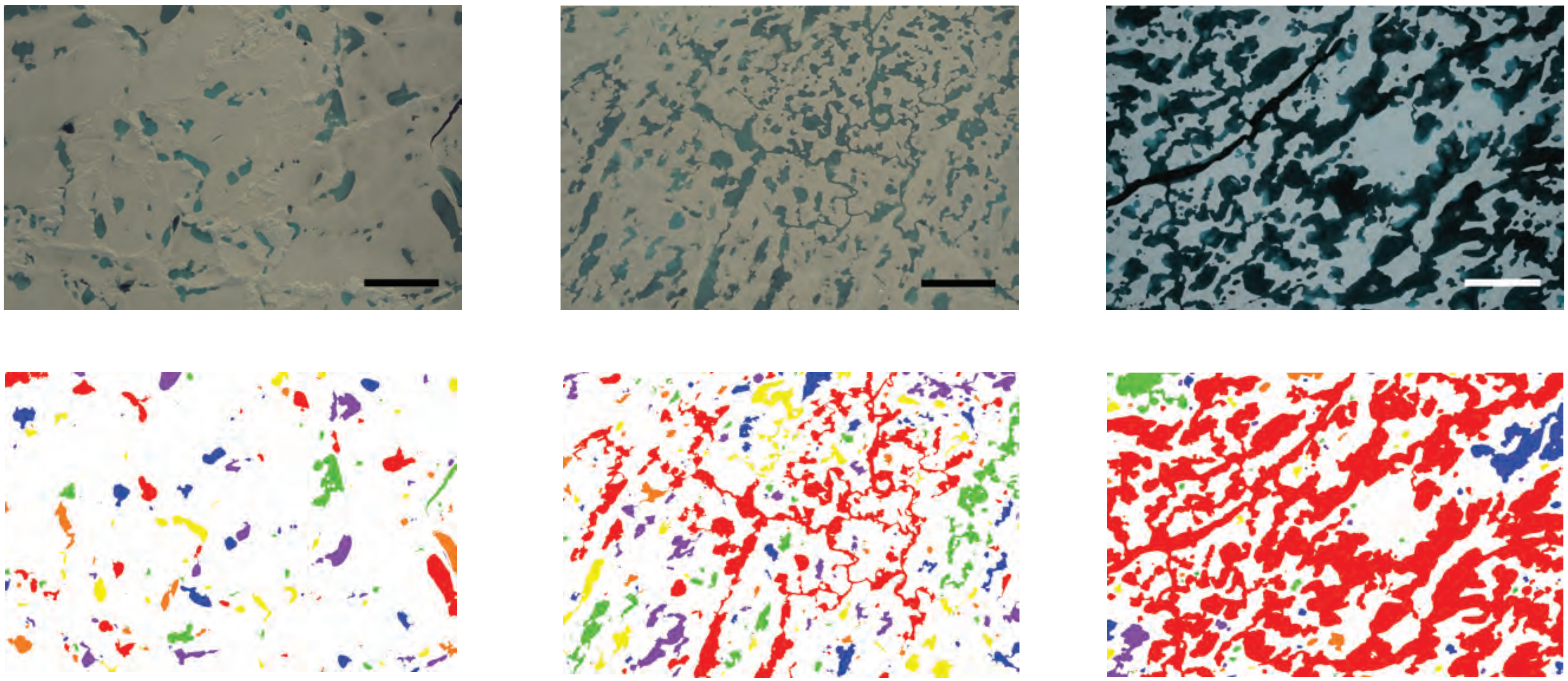
transition in the fractal dimension

complexity grows with length scale



compute "derivative" of area - perimeter data

small simple ponds coalesce to form
large connected structures with complex boundaries



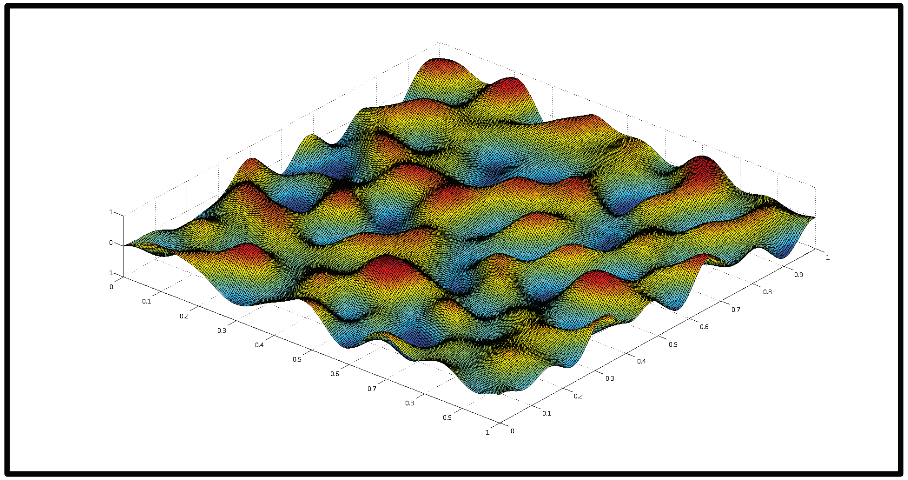
melt pond percolation

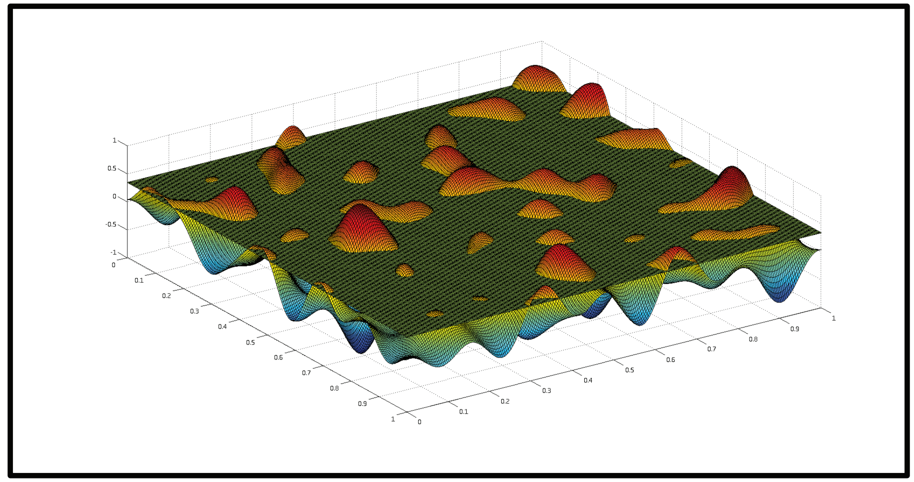
results on percolation threshold, cluster behavior

Anthony Cheng (Hillcrest HS), Bacim Alali, Ken Golden

Continuum percolation model for melt pond evolution

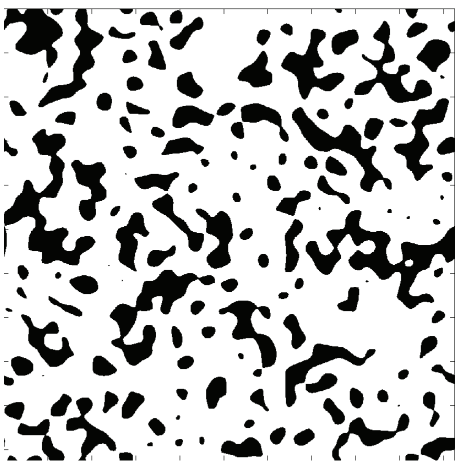
Brady Bowen, Court Strong, Ken Golden, 2015



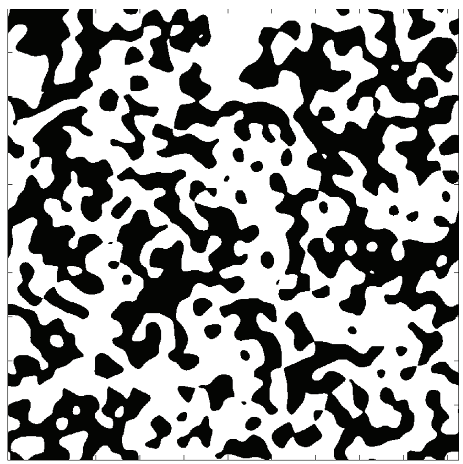


random Fourier series representation of surface topography

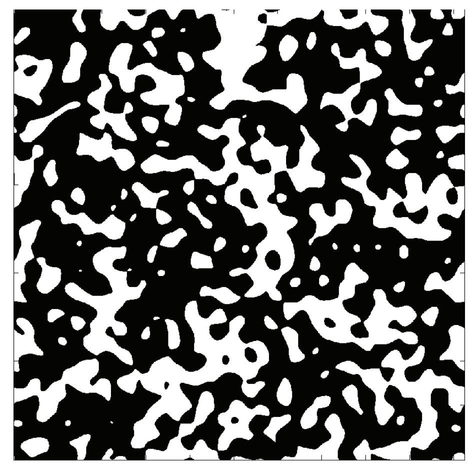
intersections of a plane with the surface define melt ponds



electronic transport in disordered media

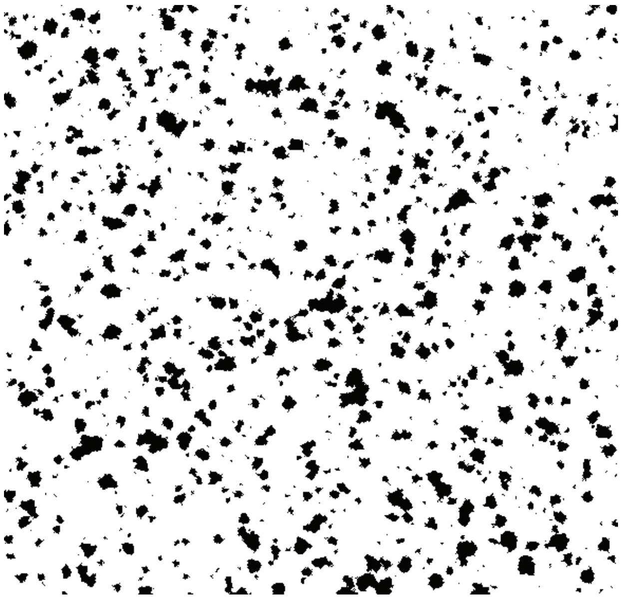


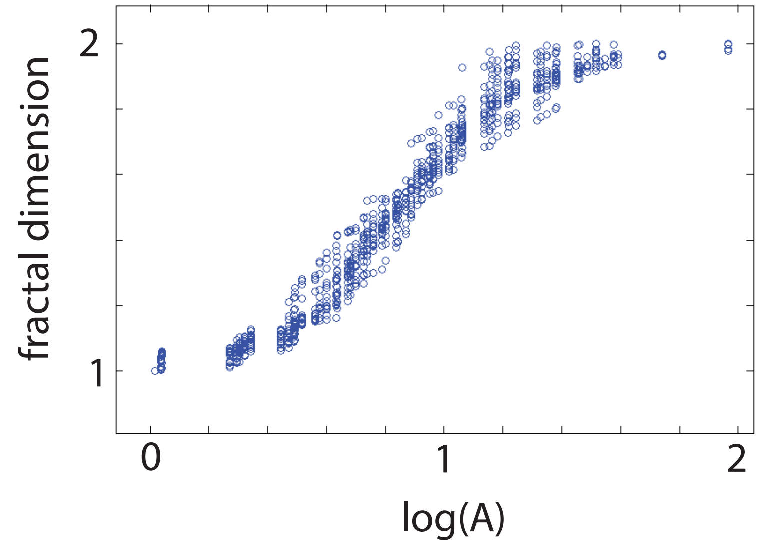
diffusion in turbulent plasmas

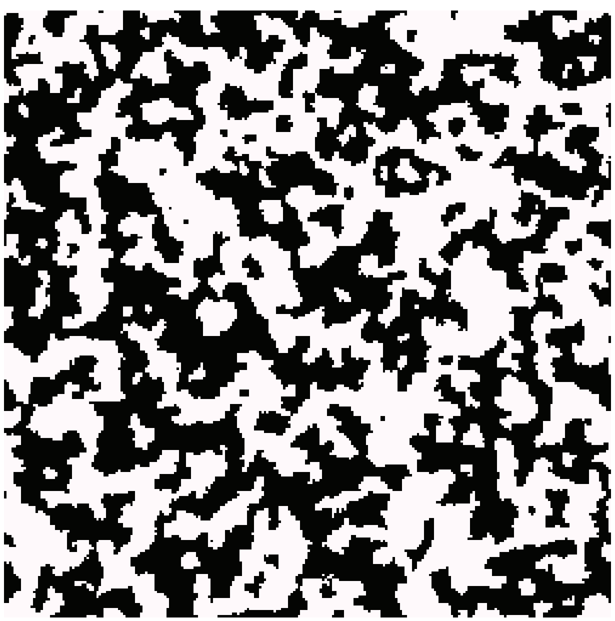


(Isichenko, Rev. Mod. Phys., 1992)

simple stochastic growth model of melt pond evolution







voter model

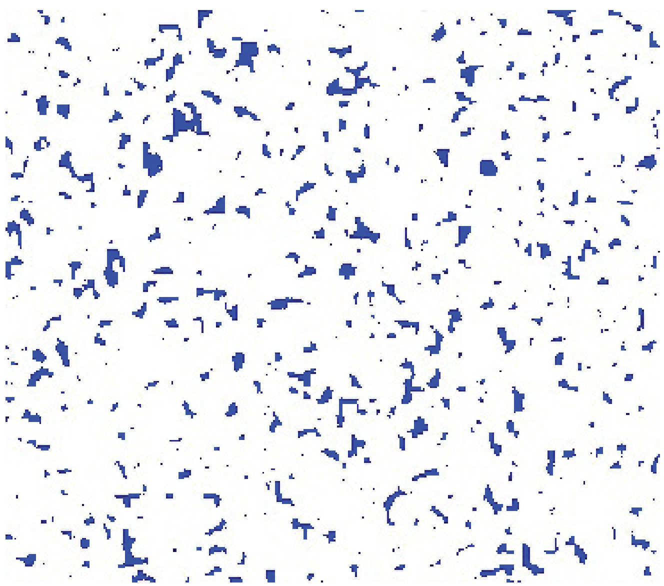
a square is more likely to melt if its neighbors have melted

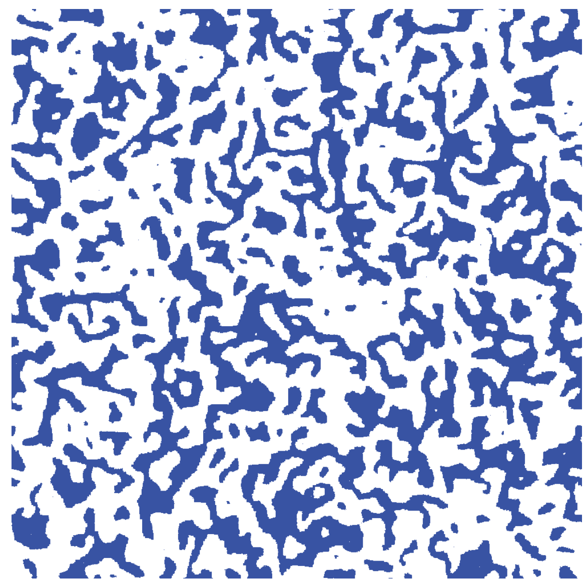
Rebecca Nickerson (West HS, Salt Lake City) and Ken Golden

Ising model for ferromagnets → Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

magnetization $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$



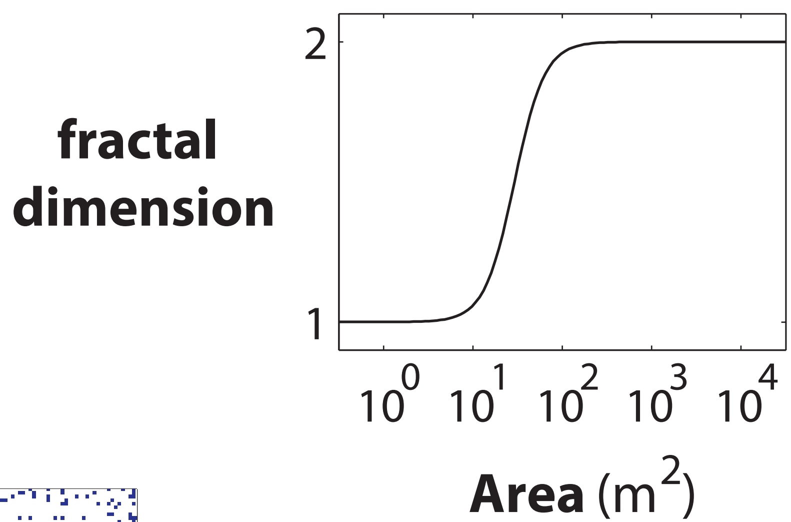


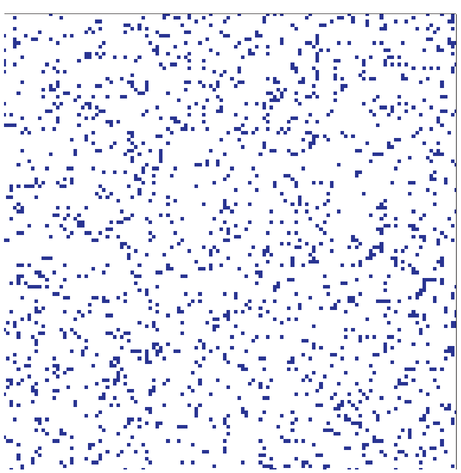
"melt ponds" are clusters of magnetic spins that align with the applied field

Ma, Sudakov, Golden 2015
(Thekkedath, Alali, Strong, Golden)

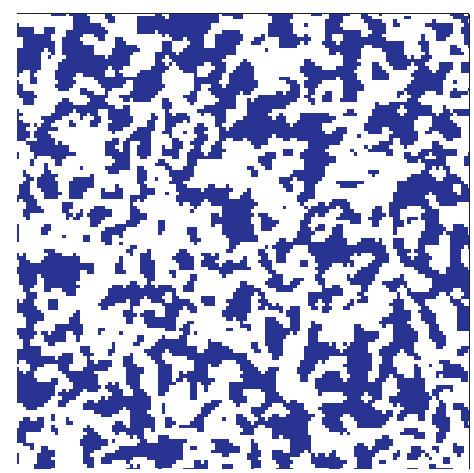
Melt Pond Ising Model

- Minimize an Ising Hamiltonian
  random magnetic field represents the initial ice topography
  interaction term represents horizontal heat transfer
- Ice-albedo feedback incorporated by taking coupling constant in interaction term to depend on the pond coverage





predicted length scale of fractal transition agrees well with data



2011 massive under-ice algal bloom

Arrigo et al., *Science* 2012

melt ponds act as

WINDOWS

allowing light through sea ice

no bloom

bloom

MELT POND CONUNDRUM

How can ponds form on highly permeable sea ice?

C. Polashenski, K. M. Golden, E. Skyllingstad, D. K. Perovich

SUBICE 2014

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Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999