# Mushy layer theory of sea ice

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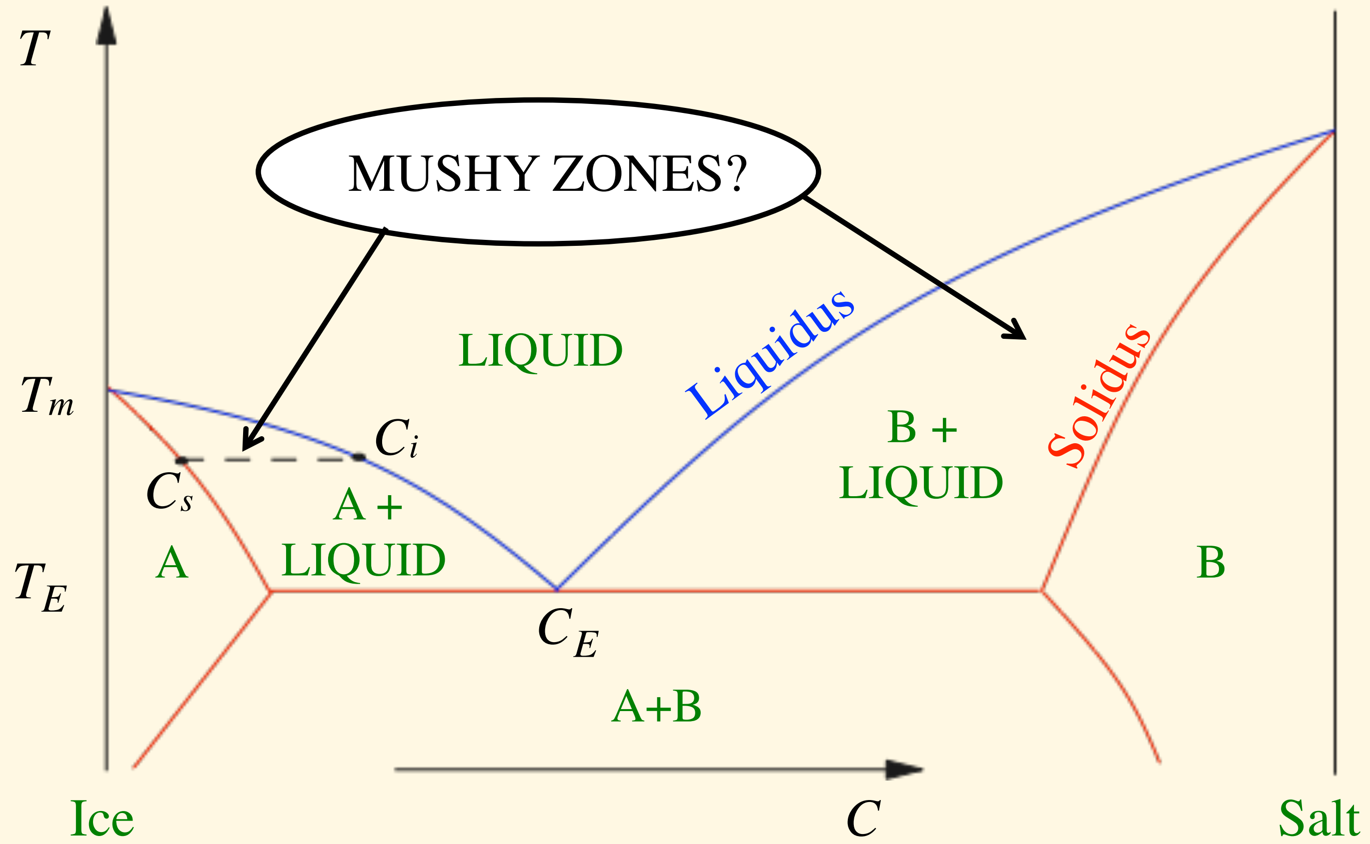
# How mushy layers form

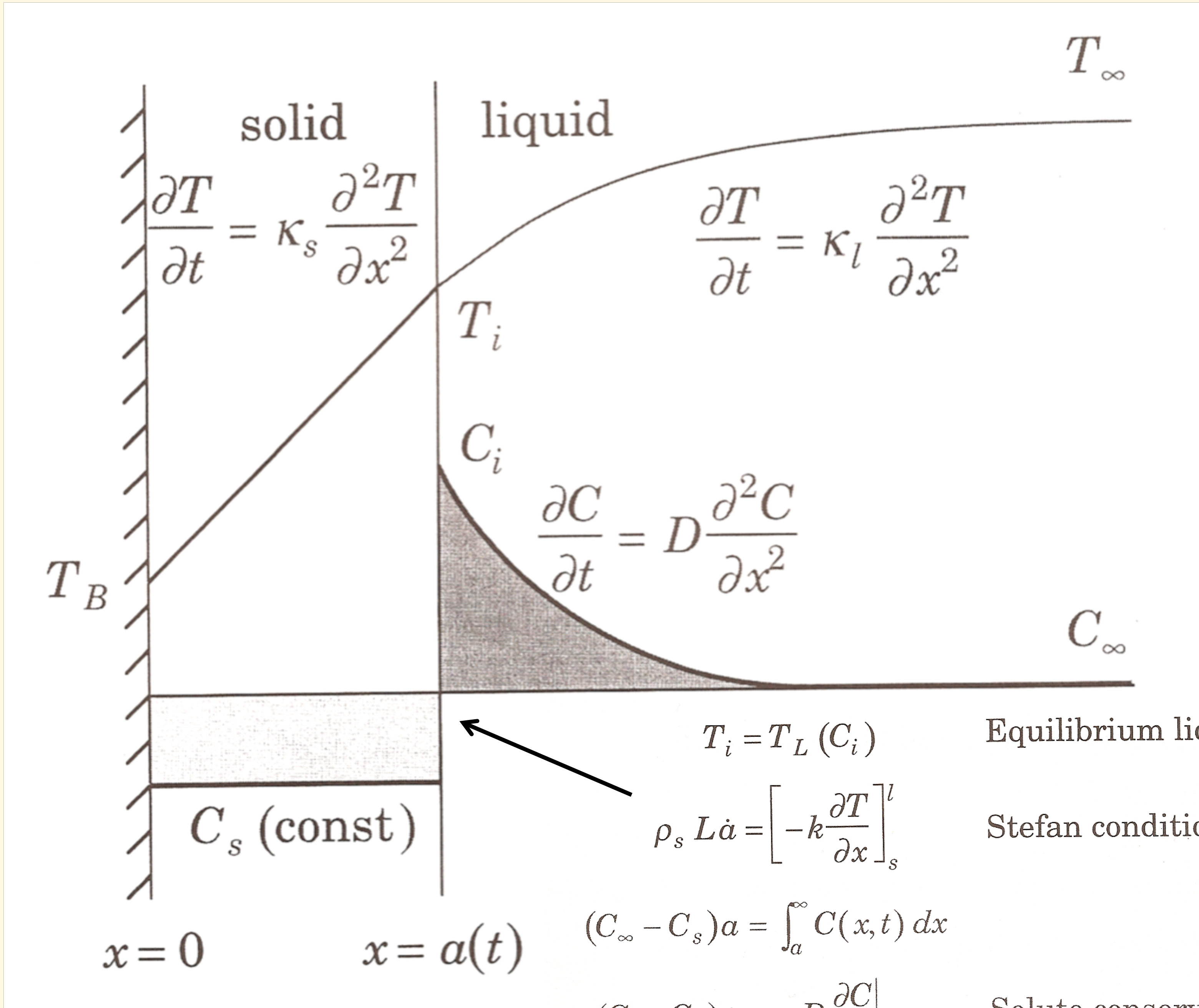
# Binary Mixtures: Cartoon Equilibrium Phase Diagram




Freezing temperature (liquidus) is a function of composition $T_L(C)$

Solid that forms has a different composition (given by the solidus) than the liquid

# Planar growth of a binary alloy, no mushy layer




$$T_i = T_L(C_i)$$ Equilibrium liquidus

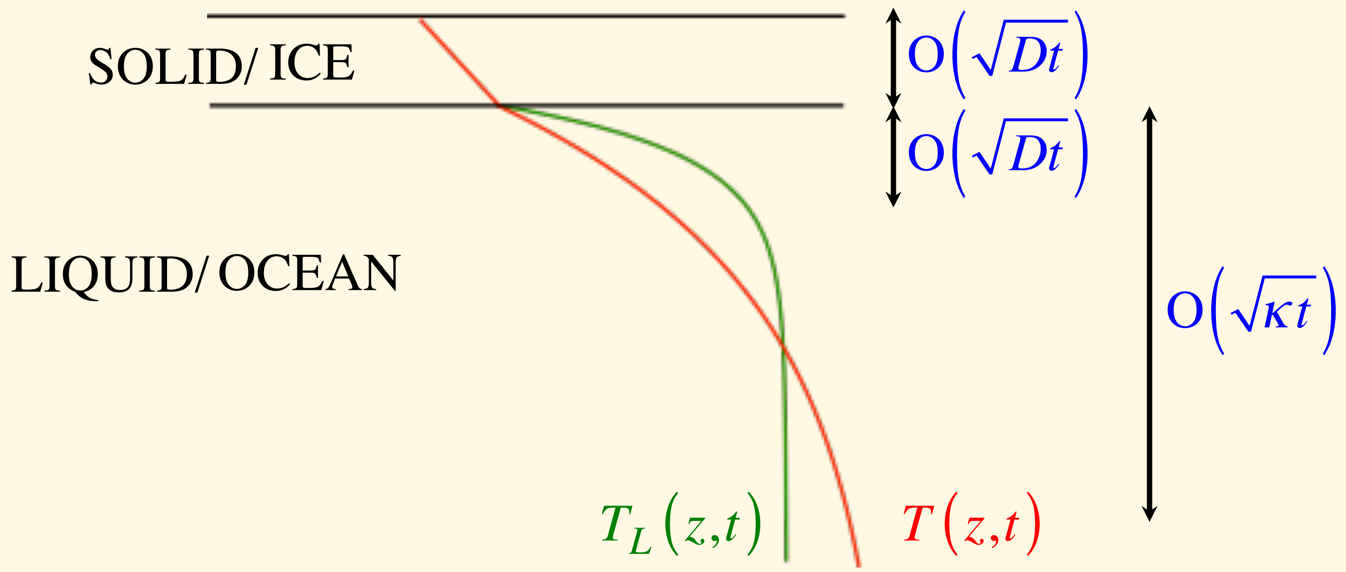
$$\rho_s L \dot{a} = \left[ -k \frac{\partial T}{\partial x} \right]_s^l$$ Stefan condition

$$(C_\infty - C_s) a = \int_a^\infty C(x,t)\, dx$$

$$\Rightarrow (C_i - C_s) \dot{a} = -D \left. \frac{\partial C}{\partial x} \right|_{a+}$$ Solute conservation

Exercise for students

# Constitutional Supercooling



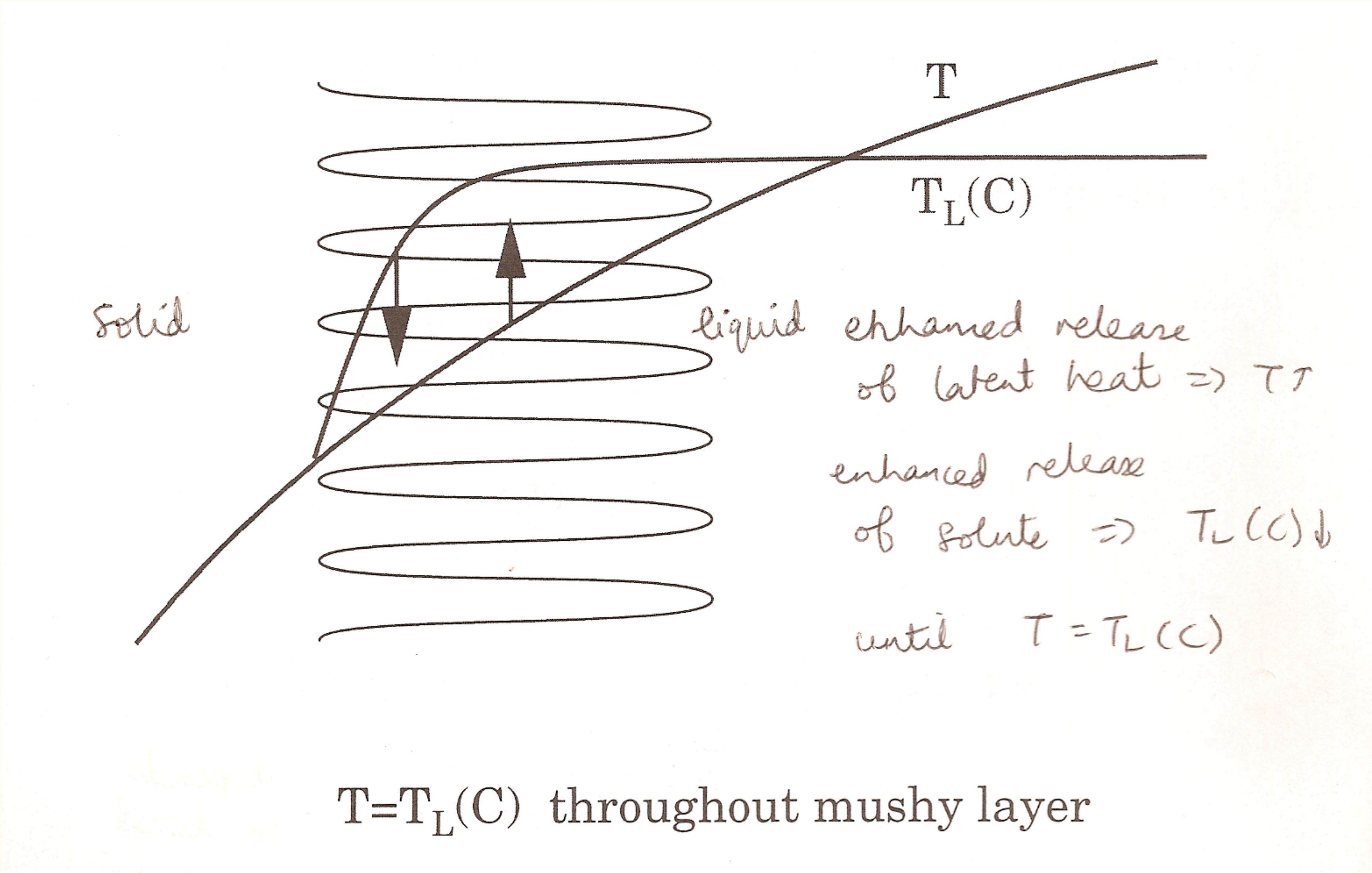

$$C = C_\infty + \left(C_i - C_\infty\right)\text{erfc}\left(z/2\sqrt{Dt}\right)/\,\text{erfc}\left(a/2\sqrt{Dt}\right)$$

$$T = T_\infty + \left(T_i - T_\infty\right)\text{erfc}\left(z/2\sqrt{\kappa t}\right)/\,\text{erfc}\left(a/2\sqrt{\kappa t}\right)$$

$$a = 2\lambda\sqrt{Dt} \qquad \sqrt{\pi}\lambda e^{\lambda^2}\,\text{erfc}\,\lambda \approx \mathcal{C}^{-1} \qquad \mathcal{C} = \frac{C_0}{\Delta C}$$

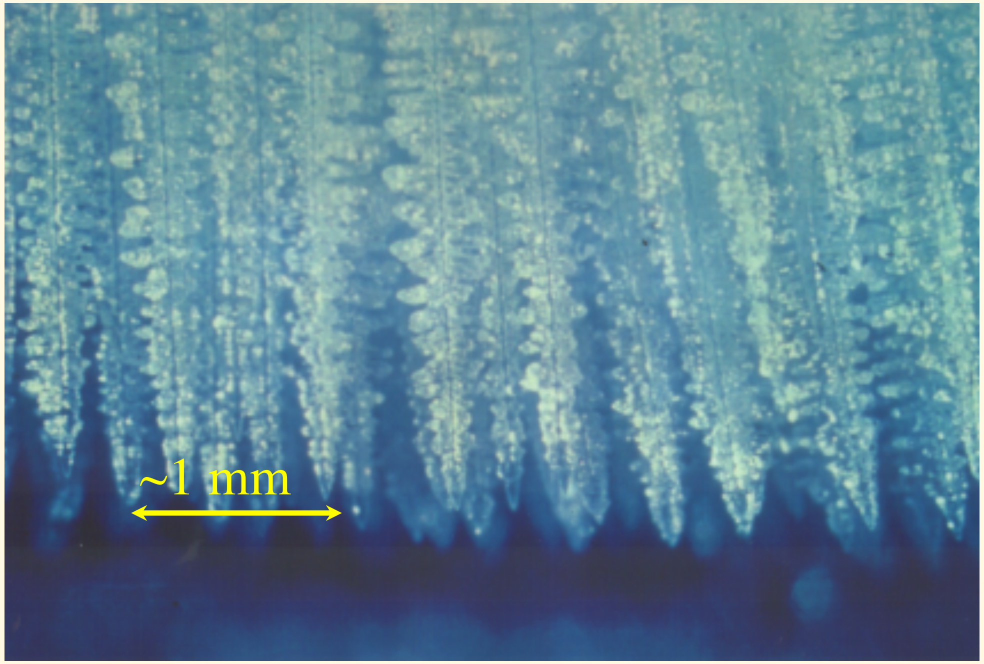
The growth rate of a planar solid–liquid interface is limited by the rate of removal of solute. Constitutional supercooling leads to the planar ice-ocean interface becoming unstable.

# Formation of a mushy layer




Corrugations of the ice-ocean interface grow to become platelets/dendrites.
The greater solid-liquid surface area results in enhanced expulsion of heat and salt until local thermodynamic equilibrium is achieved.

# Sea ice is a mushy layer





A <u>mushy layer</u> consists of a (typically) <u>porous</u> matrix of (almost) pure solid bathed in (highly concentrated) interstitial liquid.

The convoluted geometry enhances expulsion of solute and heat.

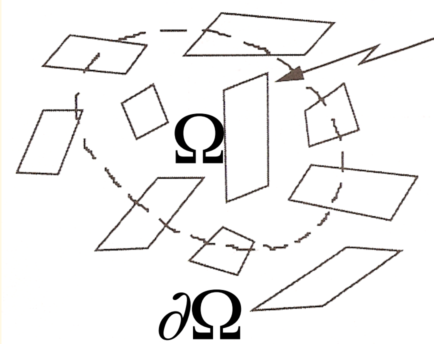
Sea ice (all of it) is an example of a mushy layer, and the mushy layer equations are used for the heat and salt balances.

# Modelling mushy layers

# Developing a continuum mushy layer model from conservation equations

Local conservation of an arbitrary scalar $\Phi$, e.g. heat, salt etc, is

$$\frac{\partial}{\partial t}\int_{\Omega}\Phi dv = -\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{J}ds + \int_{\Omega} S dv$$




where $S$ is a source/sink per unit volume and $\mathbf{J}$ is a flux of $\Phi$

$$\mathbf{J} = \mathbf{U}\Phi - D_{\Phi}\nabla\Phi \quad \text{(+ radiative flux for heat)}$$

Using the divergence theorem $\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{J}ds = \int_{\Omega}\nabla\cdot\mathbf{J}dv$

we can write, since the control volume is arbitrary and fixed, that

$$\frac{\partial\Phi}{\partial t} + (\nabla\cdot\mathbf{U})\Phi + \mathbf{U}\cdot\nabla\Phi = \nabla\cdot(D_{\Phi}\nabla\Phi) + S$$

**Conservation of mass:**

$$\frac{d}{dt}\int_V \bar{\rho}\, dV = -\int_{\partial V} \rho_l\, \boldsymbol{U}\cdot\boldsymbol{n}\, dS$$

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla\cdot(\rho_l \boldsymbol{U}) = 0$$

where $$\bar{\rho} = \varphi\rho_s + (1-\varphi)\rho_l$$

which gives

$$\boxed{\nabla\cdot\boldsymbol{U} = \left(1-\frac{\rho_s}{\rho_l}\right)\frac{\partial\varphi}{\partial t}}$$

Often approximate $\rho_s=\rho_l$, in which case we get the usual equation of continuity.

**Conservation of solute:**

$$\frac{\partial}{\partial t}\overline{\rho}\overline{C} + \nabla \cdot (\rho_l C \boldsymbol{U}) = \nabla \cdot (\overline{D} \nabla C)$$

where $$\overline{\rho}\overline{C} = \varphi \rho_s C_s + (1-\varphi)\rho_l C_l$$

which gives

$$(1-\varphi)\frac{\partial C}{\partial t} + \boldsymbol{U} \cdot \nabla C = \nabla \cdot (\overline{D} \nabla C) + \frac{\rho_s}{\rho_l}(C - C_s)\frac{\partial \varphi}{\partial t}$$

Once the liquidus constraint is imposed (see later) the diffusive term can be neglected, since $D << \kappa$

**Conservation of heat (enthalpy, *H*):**

$$\frac{\partial}{\partial t}\overline{\rho}\,\overline{H} + \nabla \cdot (\rho_l H_l \boldsymbol{U}) = \nabla \cdot (\overline{k} \nabla T)$$

where $$\overline{\rho}\,\overline{H} = \varphi \rho_s H_s + (1-\varphi)\rho_l H_l$$

which gives

$$\overline{\rho C_p}\frac{\partial T}{\partial t} + \rho_l C_{pl}\boldsymbol{U} \cdot \nabla T = \nabla \cdot (\overline{k} \nabla T) + \rho_s L \frac{\partial \varphi}{\partial t}$$

where $C_p = \frac{\partial H}{\partial T}$ and $L = H_l - H_s$

$$\overline{\rho C_p}\frac{\partial T}{\partial t} + \rho_l C_{pl}\boldsymbol{U}\cdot\nabla T = \nabla\cdot\left(\bar{k}\nabla T\right) + \rho_s L\frac{\partial \varphi}{\partial t}$$

Note that here $C_{pl}$ should strictly be taken as $C_{pl} + Q_l\, C_L{}'(T)$ where $Q_l = \dfrac{\partial H_l}{\partial C}$ is the heat of solution and $C_L(T)$ is the liquidus relationship. Note also that $L$ is a function of the temperature and composition of the liquid. Once the liquidus constraint is applied, $L$ is just a function of temperature. In many practical situations, $Q_l$ makes only a small addition to $C_{pl}$ and $L$ can be taken to be constant.

## Liquidus Relationship:

In principle , an evolution equation for the solid fraction is required., which might be of the form, for example,

$$\frac{\partial \varphi}{\partial t} = G\, \Sigma [T_L(C) - T]$$

where $G$ is a kinetic coefficient and $\Sigma$ is the specific surface area (area per unit volume) of the internal phase boundaries.

Instead, it is usually assumed that the specific surface area is sufficiently large that

$$T = T_L(C)$$

throughout the mushy layer.

# The mathematical description of a mushy layer

$$(1-\varphi)\frac{\partial C}{\partial t} + \boldsymbol{U}\cdot\nabla C = \nabla\cdot\left(\overline{D}\nabla C\right) + \frac{\rho_s}{\rho_l}\left(C - C_s\right)\frac{\partial \varphi}{\partial t}$$

$$\overline{\rho C_p}\frac{\partial T}{\partial t} + \rho_l C_{pl}\boldsymbol{U}\cdot\nabla T = \nabla\cdot\left(\overline{k}\nabla T\right) + \rho_s L\frac{\partial \varphi}{\partial t}$$

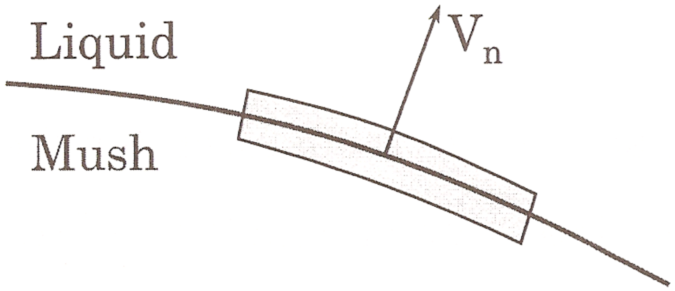
$$\nabla\cdot\boldsymbol{U} = \left(1 - \frac{\rho_s}{\rho_l}\right)\frac{\partial \varphi}{\partial t}$$

$$T = T_L(C)$$

NOTE: We have not considered conservation of momentum.
Mushy layer theory DOES NOT prescribe the interstitial fluid (brine) velocity **U**. This is determined by a local momentum balance, which is well approximated by Darcy's Law in a porous medium

$$\mathbf{U} = -\frac{\Pi}{\mu}\nabla p'$$

**Interfacial conditions**




In a frame of reference moving with the interface

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - V_n\, \boldsymbol{n}\cdot\nabla$$

Integrate the governing equations, in conservative form, over the control volume shown:

Mass: $$[U\cdot n]_m^l = -V_n\left(1-\frac{\rho_s}{\rho_l}\right)[\varphi]_m^l$$

Solute: $$-V_n\left[\overline{\rho}\overline{C}\right]_m^l + \rho_l C[\boldsymbol{U}\cdot\boldsymbol{n}]_m^l = \left[\overline{D}\boldsymbol{n}\cdot\nabla C\right]_m^l$$

$$\Rightarrow \quad \rho_s(C_s - C)\varphi V_n = D\frac{\partial C}{\partial n}\bigg|_l - \overline{D}\frac{\partial C}{\partial n}\bigg|_m$$

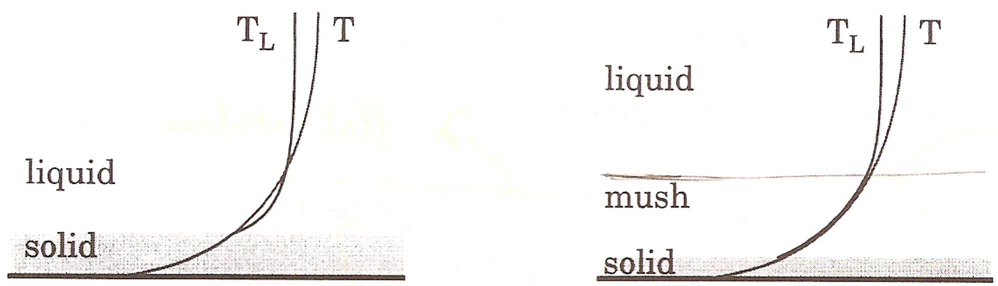
Heat: $$-V_n\left[\overline{\rho}\overline{H}\right]_m^l + \rho_l H_l[\boldsymbol{U}\cdot\boldsymbol{n}]_m^l = \left[\overline{k}\boldsymbol{n}\cdot\nabla T\right]_m^l$$

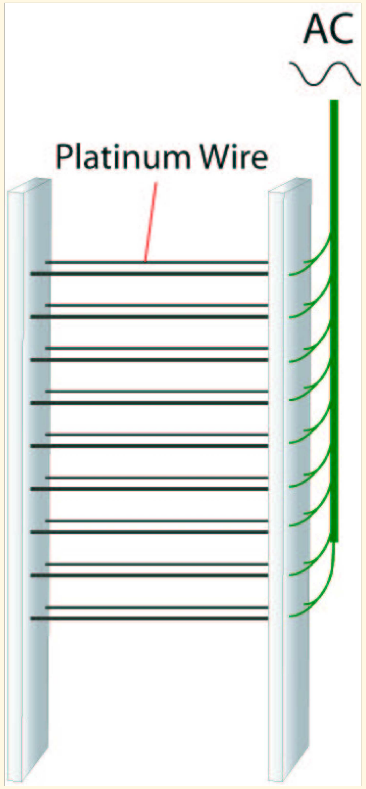
$$\Rightarrow \quad \rho_s L\varphi V_n = k_l\frac{\partial T}{\partial n}\bigg|_l - \overline{k}\frac{\partial T}{\partial n}\bigg|_m$$

## Marginal equilbrium

Notz, Wettlaufer and Worster [2005]

Consistent with the approach used in the interior of the mushy layer, apply a condition of "marginal equilibrium" that the mushy layer grows sufficiently to eliminate undercooling near the interface.

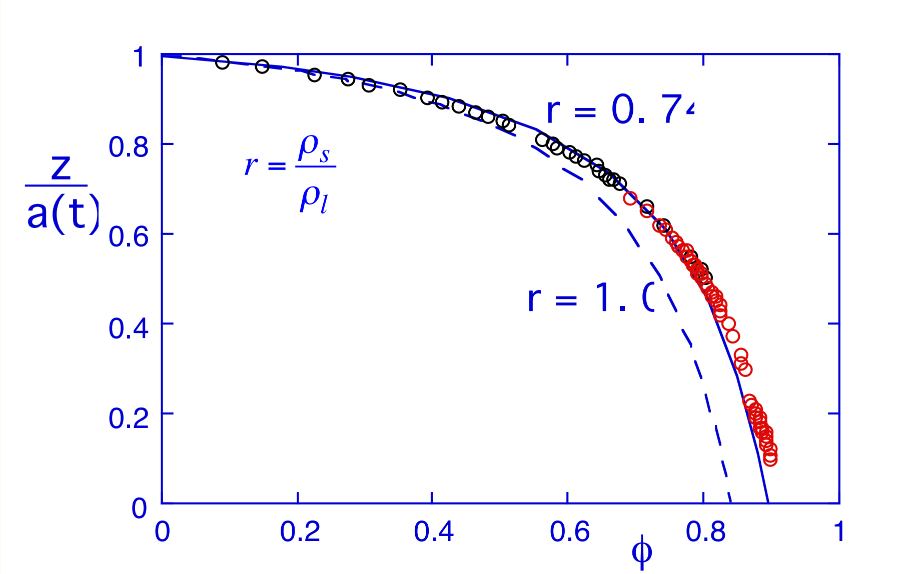



This translates to the condition

$$\left.\frac{\partial T}{\partial n}\right|_l = \left.\frac{\partial T_L}{\partial n}\right|_l$$

Note that in many (most) circumstances this condition implies that $\phi = 0$ at the mush–liquid interface.



# Mushy layer model of sea ice with no brine flow

$$\overline{\rho c_p}\frac{\partial T}{\partial t} = \nabla\cdot\left(\bar{k}\nabla T\right) + \rho_s L\frac{\partial \varphi}{\partial t}$$

The local bulk concentration is $C_B = (1-\varphi)C$ so $\varphi = 1 - \frac{C_B}{C}$

The interstitial concentration $C$ is related to the local temperature $T$ by the liquidus relation

$$T = T_L(C) \equiv T_m - mC$$

Then if $\frac{\partial C_B}{\partial t} \equiv 0$, so that $C_B = C_B(\mathbf{x})$ only,

$$\left[\overline{\rho c_p} + \rho_s L\frac{mC_B}{(T_m - T)^2}\right]\frac{\partial T}{\partial t} = \nabla\cdot\left(\bar{k}\nabla T\right)$$

The thermal inertia (specific heat capacity) of sea ice is dominated by the internal release of latent heat.

**The mushy layer equations with <u>no brine flow</u> are essentially the same as the Maykut and Untersteiner [1971] model of sea ice.** In recent years, mushy layer theory has been used to model desalination of sea ice, morphological instability, melt pond evolution, nutrient dynamics, consolidation of rafted sea ice, etc.

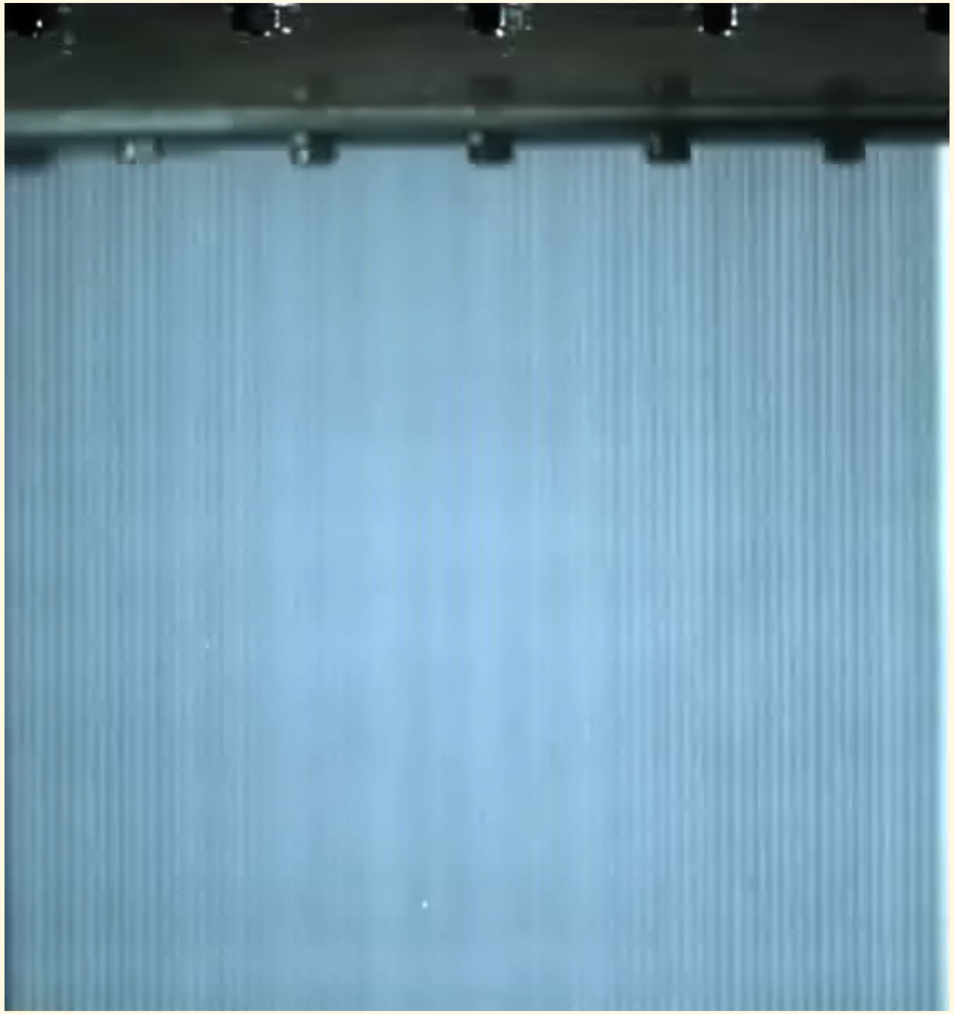
# Brine flow in sea ice:

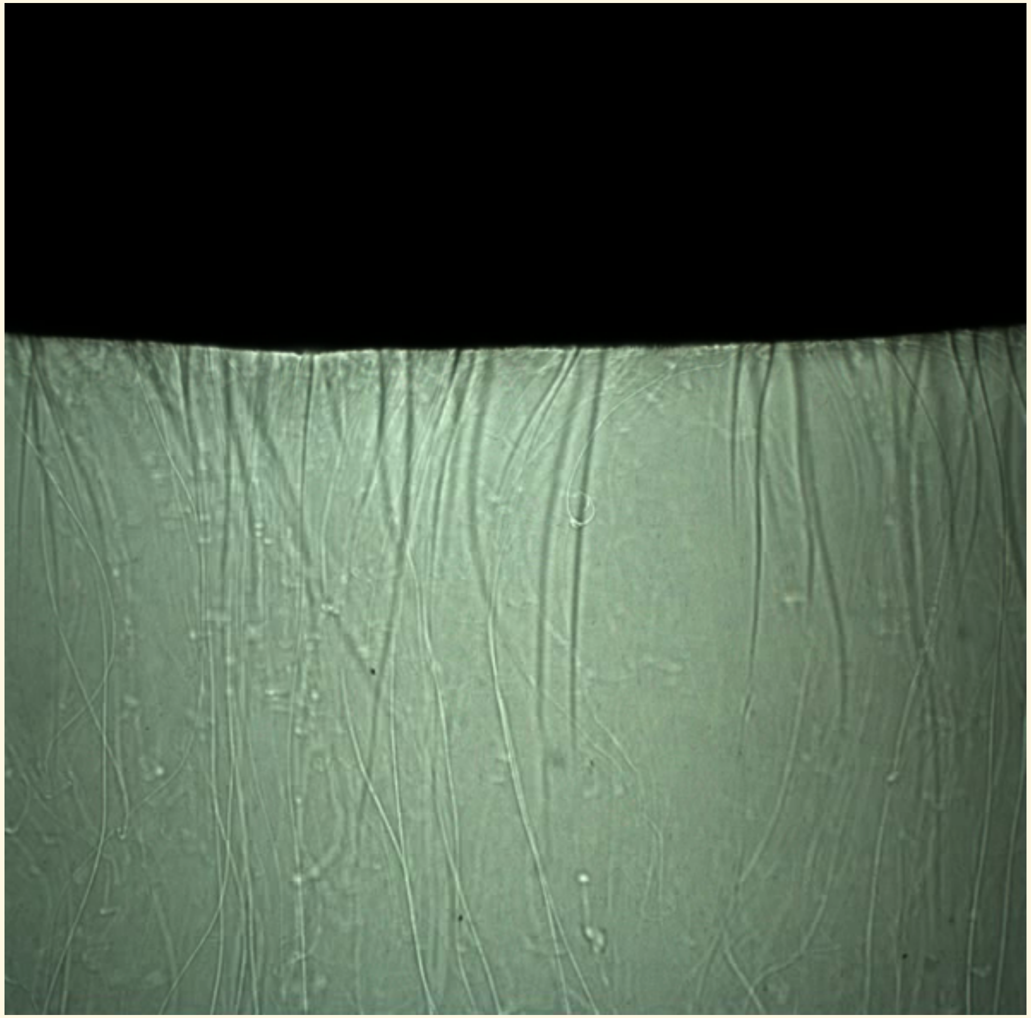
## condition for onset of buoyancy-driven convection

(*not* flushing)

# Brine Drainage from Sea Ice

Wettlaufer, Worster, and Huppert [1997]

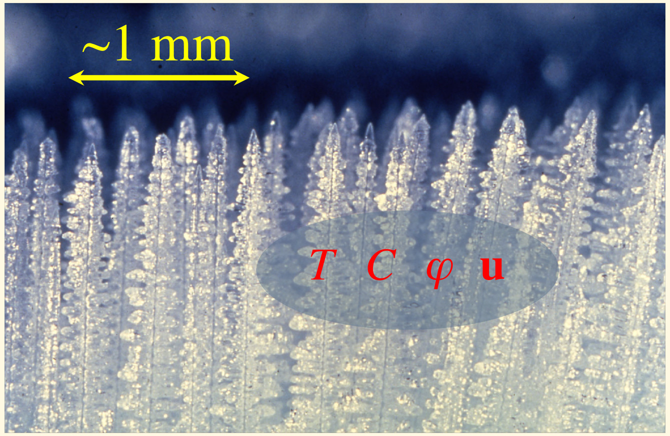




Unstable density profile of brine salinity in sea ice can drive so-called “gravity drainage”.

# Linear stability analysis of a mushy layer - onset of convection (brine drainage)

adapted from Worster 1992




Methodology:
perturb a steady state and calculate if perturbations grow

'Ideal mushy layer'— thermal properties constant and independent of phase and $\rho_s = \rho_l$

Conservation of heat $$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + \frac{L}{c_p} \frac{\partial \varphi}{\partial t}$$

Conservation of salt $$(1-\varphi)\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = C \frac{\partial \varphi}{\partial t}$$ ignore diffusion of salt

Liquidus constraint $$T = T_L(C) = T_0 - m(C - C_0)$$

Darcy's equation
for flow in a porous medium $$\mu \mathbf{u} = \Pi[-\nabla p + \rho \mathbf{g}]$$

Incompressibility $$\nabla \cdot \mathbf{u} = 0$$

Consititutive relation $$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(C - C_0)] = \rho_0[1 + \beta^*(C - C_0)]$$

# Non-dimensionalising and scaling

Suppose that system is solidifying at rate $V$ and scale

velocities with $V$ lengths with $\kappa / V$ time with $\kappa / V^2$ pressure with with $\Delta\rho g\kappa / V$

Write $\theta = \frac{T - T_0}{\Delta T} = -\frac{C - C_0}{\Delta C}$ where $\Delta T = m\Delta C$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta + S\frac{\partial\varphi}{\partial t}$$

$$(1-\varphi)\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = (\theta - \mathcal{C})\frac{\partial\varphi}{\partial t}$$

$$\mathbf{u} = R_m\left[-\nabla p + \theta\mathbf{k}\right]$$

$$\nabla\cdot\mathbf{u} = 0$$

$S \gg 1,\ \mathcal{C} \gg 1$ → $\frac{S}{\mathcal{C}} = \mathrm{O}(1)$

$$\Omega\left(\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta\right) = \nabla^2\theta$$

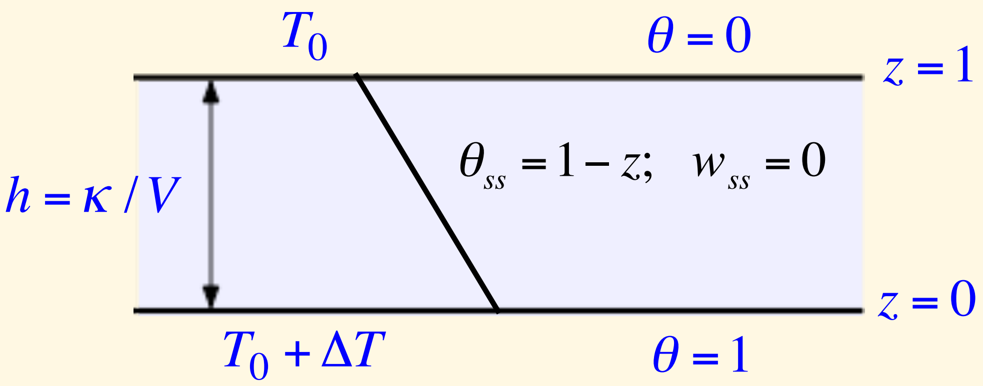
$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -\mathcal{C}\frac{\partial\varphi}{\partial t}$$

$$\mathbf{u} = R_m\left[-\nabla p + \theta\mathbf{k}\right]$$

$$\nabla\cdot\mathbf{u} = 0$$

$$S = \frac{L}{c_p\Delta T} \qquad \mathcal{C} = \frac{C_0}{\Delta C} \qquad \Omega = 1 + \frac{S}{\mathcal{C}} \qquad R_m = \frac{\beta^*\Delta C g\Pi(\kappa/V)}{\kappa\nu}$$

# Indicative Stability Analysis




$$\Omega\left(\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta\right) = \nabla^2\theta$$

$$\mathbf{u} = R_m\left[-\nabla p + \theta\mathbf{k}\right]$$

$$\nabla\cdot\mathbf{u} = 0$$

Write $\mathbf{u} = (u, w)$

$$\theta = 1 - z + \hat{\theta}(z)e^{i\alpha x + \sigma t}$$

$$w = \quad \hat{w}(z)e^{i\alpha x + \sigma t}$$

steady state ↗ ↖ perturbations

σ is the growth rate
α is the wavenumber

Substitute, linearize and look for marginal (steady) states with $\sigma = 0$

$$D \equiv \frac{d}{dz}$$

$$-\Omega\hat{w} = \left(D^2 - \alpha^2\right)\hat{\theta}$$

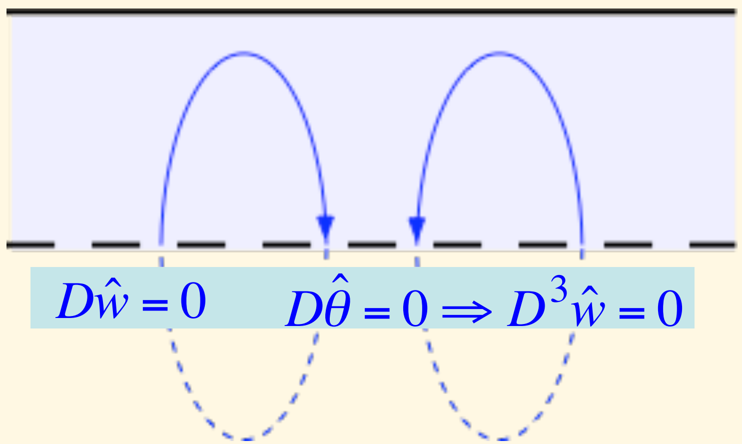
$$\left(D^2 - \alpha^2\right)\hat{w} = \alpha^2 R_m\hat{\theta}$$

$$\Rightarrow \quad \left(D^2 - \alpha^2\right)^2\hat{w} = -\alpha^2\Omega R_m\hat{w}$$

# Marginal Stability Results

$$\left(D^2-\alpha^2\right)^2\hat{w}=-\alpha^2\Omega R_m\hat{w}$$

No flow, constant temperature $\quad \hat{w}=0 \qquad \hat{\theta}=0\Rightarrow D^2\hat{w}=0$



Constant pressure and heat flux $\quad D\hat{w}=0 \qquad D\hat{\theta}=0\Rightarrow D^3\hat{w}=0$

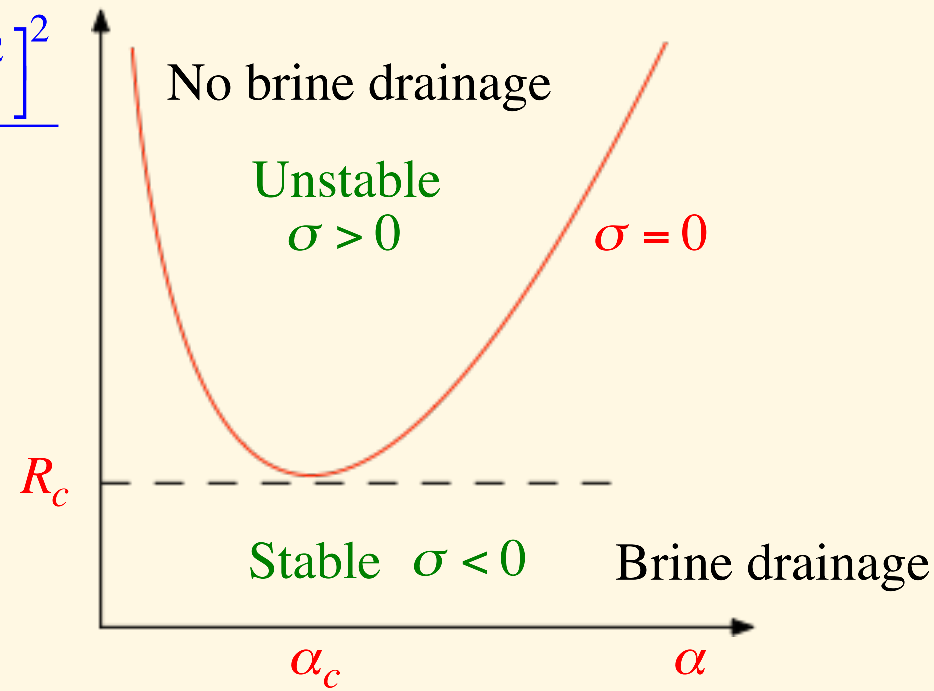
$$\hat{w}\propto\sin\left[\left(n+\tfrac{1}{2}\right)\pi\left(1-z\right)\right] \qquad \Omega R_m=\frac{\left[\left(n+\frac{1}{2}\right)^2\pi^2+\alpha^2\right]^2}{\alpha^2}$$

Lowest (most unstable) mode has $n=0$ i.e. the circulation cell is of the same size as the mushy layer depth

$$\Omega R_m=\frac{\left[\frac{\pi^2}{4}+\alpha^2\right]^2}{\alpha^2}$$

# Critical Rayleigh Number

$$\Omega R_m = \frac{\left[\frac{\pi^2}{4} + \alpha^2\right]^2}{\alpha^2}$$



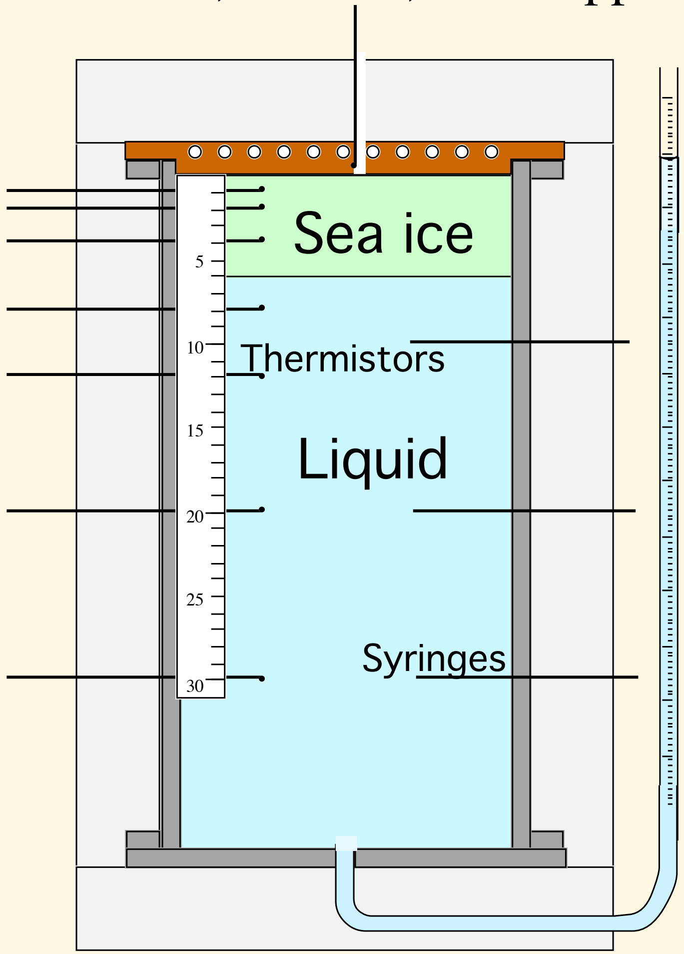

"most unstable" wavenumber is $\alpha_c = \frac{\pi}{2}$

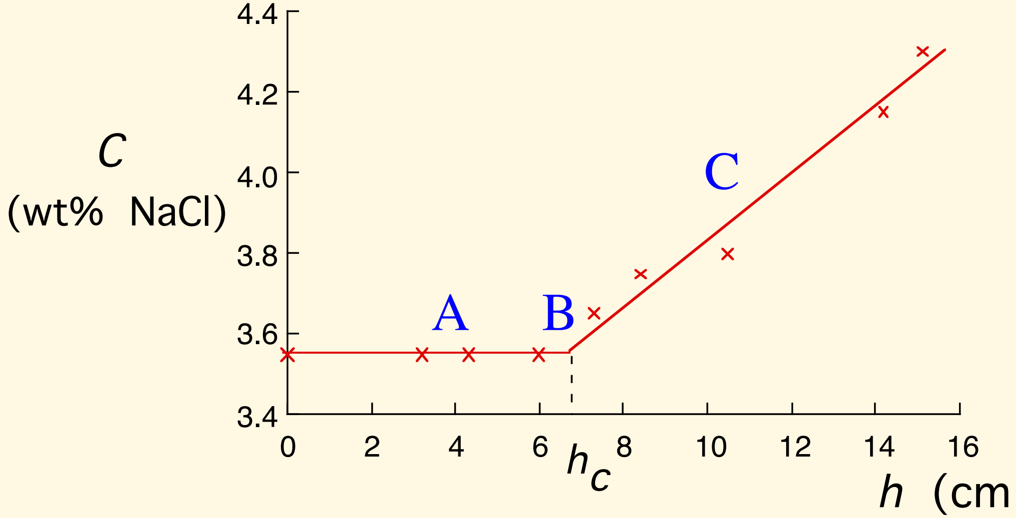
where the critical Rayleigh number is $\Omega R_m > R_c = \pi^2 \approx 10$

Brine drainage will not occur until the critical Rayleigh number is reached.

# Brine Drainage from Laboratory Sea Ice
# Stages of Evolution

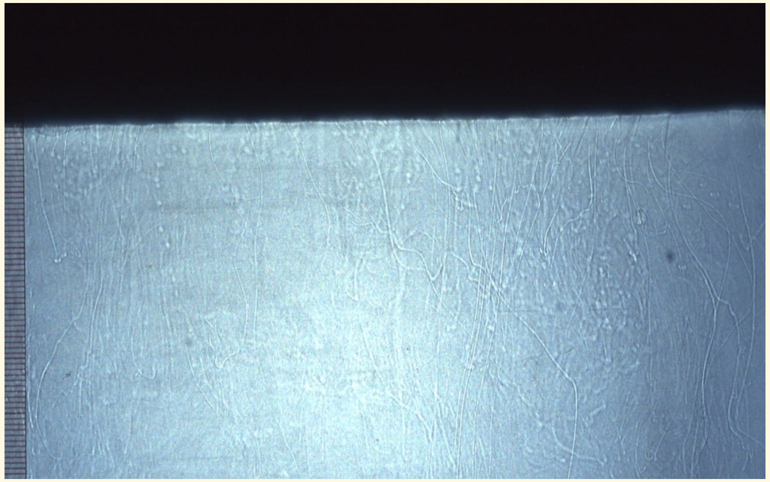
Wettlaufer, Worster, and Huppert [1997]



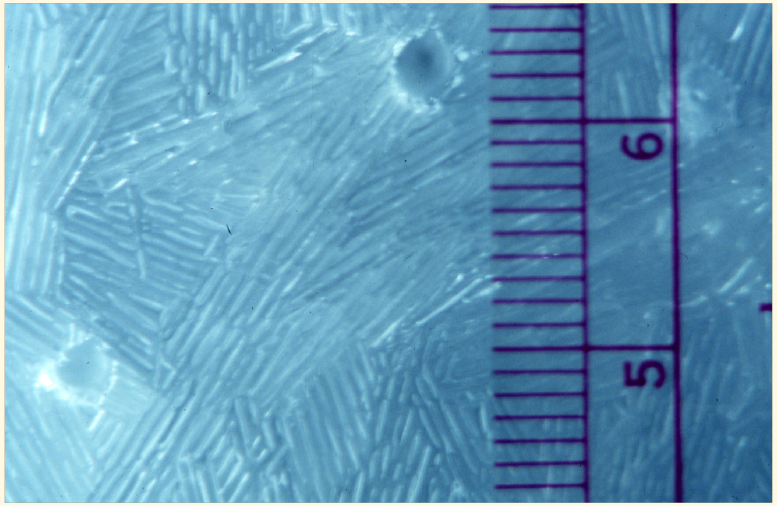




A: Quiescent
B: Transition
C: Draining from brine channels

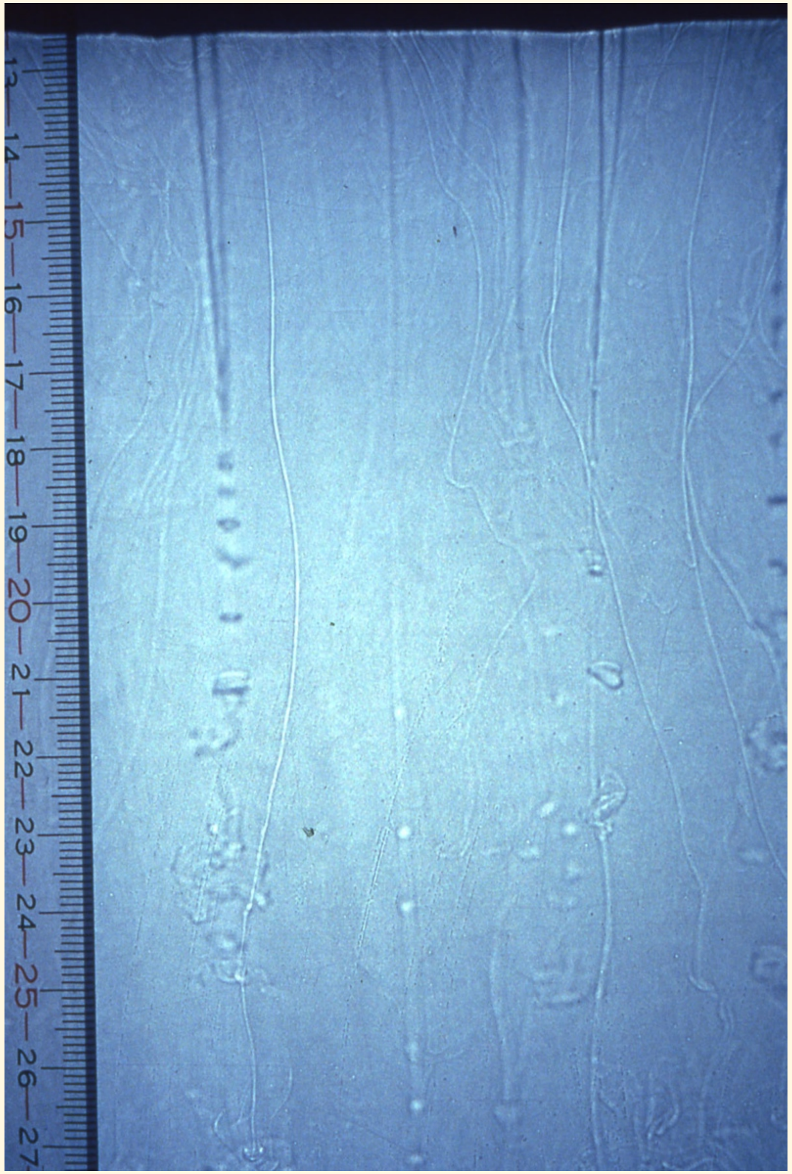
Stages of Drainage from Sea Ice



A: Quiescent




Surface expression of brine channels



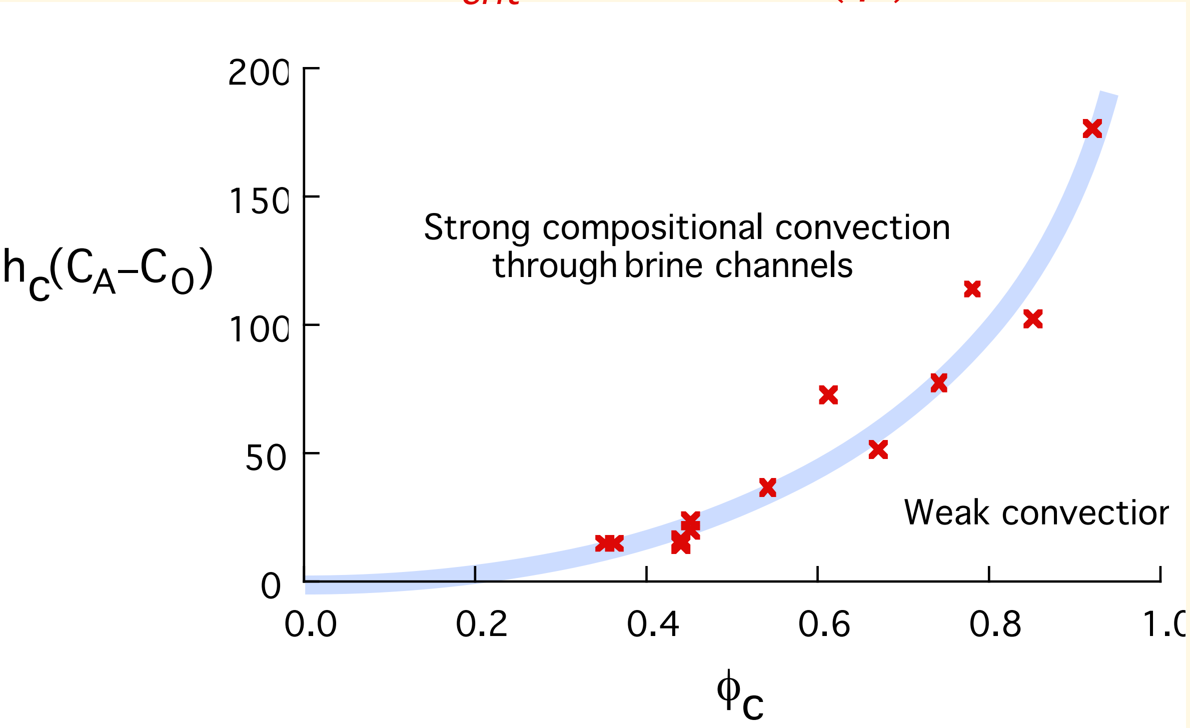

C: Draining

# Onset of Brine Drainage

Convection is determined by a critical value of a Rayleigh number

$$R_m = \frac{\beta \Delta C g \Pi h}{\kappa \nu} = R_{crit}$$

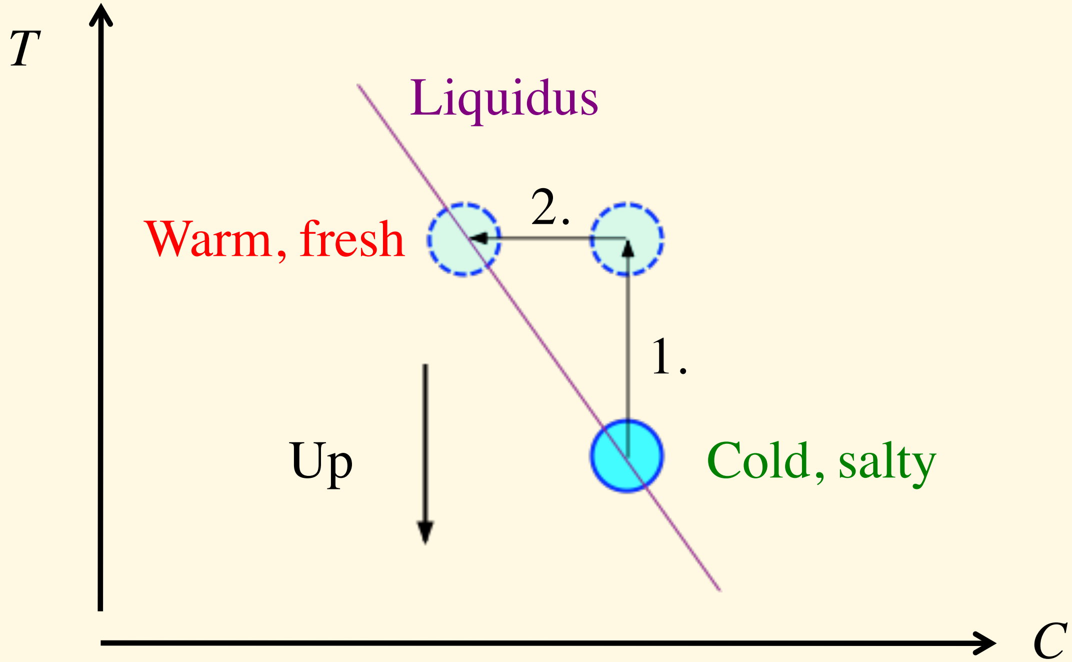
Thus $$h_{crit} \Delta C \propto \Pi^{-1}(\phi)$$




NB $R_{crit} = 10$ gives a permeability $\Pi \approx 10^{-10}$ m$^2$ at $\varphi$=0.8

# Understanding the Critical Condition

$$\Omega R_m = \left(1 + \frac{S}{C}\right)\frac{\beta^* \Delta C g \Pi h}{\kappa \nu} > R_c$$




A parcel of interstitial brine displaced downwards:

1: Comes to thermal equilibrium with its new surroundings by diffusion

2: Comes to complete equilibrium (liquidus) by *dissolving* the ice matrix

The driving force for convection is the unstable density profile. However, once the fluid starts moving its buoyancy is dissipated by thermal diffusion and its kinetic energy is dissipated by friction (viscosity).

# Summary

Rate of solidification of a mixture at a flat solid–liquid interface is limited by solute diffusivity.

Rejected solute causes local constitutional supercooling and morphological instability ...

... leading to the development of a mushy layer.

Sea ice is a mushy layer. Local conservation equations describe a mushy layer.

Brine fluxes arise due to the unstable buoyancy profile of brine in sea ice.

Convection in a mushy layer seems to be confined to a region where the porous medium Rayleigh number is close to the critical value of about 10.

Convection in sea ice causes formation of brine channels by dissolution.