EXAM III SOLUTIONS

1. (a) $f(x) = \dfrac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$, radius=1. The singularities of $f(x)$ are at $\pm i$, which are both one unit away from the origin.

   (b) $\tan^{-1} x = \int_0^x \dfrac{1}{1+t^2}dt = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k}dt = \sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1}$, radius=1.

2. (a) The series converges on the interval $(2,4)$, diverges when $x=4$ and converges conditionally when $x=2$.

   (b) The series converges on the interval $\left(-\frac{2}{1+\sqrt{5}}, \frac{2}{1+\sqrt{5}}\right)$.

3. For $|x| < 1$, the geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, so for the same x, the series $\sum_{n=1}^{\infty} x^n = \dfrac{x}{1-x}$.

4. (a) Converges conditionally (alternating series test, then integral test to show failure of absolute convergence)

   (b) Converges since as $n \to \infty$ we have $\frac{1}{n} \to 0$ so for large n $\sin^2\left(\frac{1}{n}\right) \sim \frac{1}{n^2}$ for which the series would converge by the p-series test.

   (c) Converges (ratio test) since $\lim\limits_{n\to\infty} \frac{(n+1)^{100}}{n^{100}(n+1)} = 0$

   (d) Diverges (n^{th} term test) since the $\lim\limits_{n\to\infty}(1+\dfrac{1}{n})^n = e \neq 0$

5. $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$, $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$, and we have $\frac{dy}{dx} = y$ and the initial condition $y(0) = a_0 = 1$ setting the two series equal to each other and matching coefficients gives:

$$a_1 = a_0 = 1 \quad a_2 = \tfrac{a_1}{2} = \tfrac{1}{2}$$
$$a_3 = \tfrac{a_2}{3} = \tfrac{1}{6} \quad a_4 = \tfrac{a_3}{4} = \tfrac{1}{24}$$

and so on, so $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots = e^x$.