EXAM II SOLUTIONS Spring 2013

1. (a) $\displaystyle\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1} = \tan^{-1}(x+3)+C.$

   (b)
$$\begin{aligned}\int \frac{2x+1}{x^2-1}\,dx &= \int \frac{1}{2(x+1)}\,dx + \int \frac{3}{2(x-1)}\,dx \\ &= \frac{1}{2}\ln(x+1) + \frac{3}{2}\ln(x-1) + C.\end{aligned}$$

   (c) Let $u = e^x$, then $du = e^x dx$ so
$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{1}{1+u^2}du = tan^{-1}(u) + c = tan^{-1}(e^x) + c.$$

   (d) $\int e^x sin(x) dx$ Integrate by parts twice. Let $u = sin(x)$ $dv = e^x$ then $du = cos(x)$ and $v = e^x$
$$\int e^x sin(x)dx = e^x sin(x) - \int e^x cos(x)dx$$
   Let $u = cos(x)$ and $dv = e^x$ then $du = -sin(x)$ and $v = e^x$ Then:
$$\int e^x sin(x)dx = e^x sin(x) - \int e^x cos(x)dx = e^x sin(x) - (e^x cos(x) - \int e^x(-sin(x))dx)$$
$$\int e^x sin(x)dx = e^x sin(x) - e^x cos(x) - \int e^x sin(x)dx$$
$$2\int e^x sin(x)dx = e^x sin(x) - e^x cos(x)$$
$$\int e^x sin(x)dx = \frac{1}{2}(e^x sin(x) - e^x cos(x))$$

   (e) $\int_1^e ln(x)dx$ Integrate by parts: $Let u = ln(x)$ and $dv = dx$ then $du = \frac{1}{x}$ and $v = x$
$$\int_1^e ln(x)dx = xln(x)|_1^e - \int_1^e x\frac{1}{x}dx$$
$$\int_1^e ln(x)dx = xln(x)|_1^e - \int_1^e dx$$
$$\int_1^e ln(x)dx = xln(x)|_1^e - x|_1^e$$
$$eln(e) - 1ln(1) - (e-1) = 1$$

2. $\int_0^b xe^{-x^2}\,dx$ This integral converges since $\int \frac{x}{e^{x^2}} \leq \int \frac{x}{x^n}$ for any n choose $n=3$ Then $\int_0^\infty \frac{x}{e^{x^2}} \leq \int_0^\infty \frac{x}{x^3} = \int_0^\infty \frac{1}{x^2}$ Which converges by the p-test. Let $u = x^2$ Then $du = 2xdx$ and we have:

$$\frac{1}{2}\int_0^\infty e^{-u}du = \lim_{b\to\infty} \frac{-1}{2}(e^{-b} - e^0) = \frac{1}{2}$$

3. (a) $\int_{-1}^1 \frac{1}{x^2}\,dx = 2\int_0^1 \frac{1}{x^2}\,dx$ diverges since the power of the x is greater than 1.

   (b) The function $\frac{1}{x(1-x)}$ goes to 0 as x goes to infinity like $\frac{1}{x^2}$, but goes to infinity as x goes to 1 like $\frac{1}{1-x}$, therefore the integral diverges.
   $\int_1^\infty \frac{dx}{x(1-x)} = \int_1^\infty \frac{1}{x} + \frac{1}{1-x}\,dx = \ln\left(\frac{x}{1-x}\right)|_1^\infty = \lim_{x\to\infty} \ln\left(\frac{x}{1-x}\right) - \lim_{x\to 1} \ln\left(\frac{x}{1-x}\right) = -\infty$

   (c) $\int \sin x\,dx = \cos x$ and $\lim_{x\to\infty} \cos x$ does not exist so the integral diverges.

   (d) $\int_1^\infty \frac{\sqrt{x+\pi^\pi}}{(1000x^4 + 37x^3 + 16x^2 + x + e)^{(1/3)}} dx$ is asymptotic to $\int_1^\infty \frac{x^{1/2}}{(1000x^4)^{1/3}} = \frac{1}{10}\int_1^\infty \frac{x^{1/2}}{x^{4/3}} = \frac{1}{10}\int_1^\infty x^{1/2-4/3} = \frac{1}{10}\int_1^\infty x^{3/6-8/6} = \frac{1}{10}\int_1^\infty \frac{1}{x^{5/6}}$
   Which diverges by the p-test.

4. (a) By L'Hopital's rule,

$$\lim_{x\to 1} \frac{\ln x - x + 1}{x^3 - 3x + 2} = \lim_{x\to 1} \frac{\frac{1}{x} - 1}{3x^2 - 3} = \lim_{x\to 1} \frac{-1/x^2}{6x} = -\frac{1}{6}.$$

   (b) We apply L'Hopital's rule to

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

   (c) By L'Hopital's rule,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

   We can interperet this in the following way: Near zero $\sin(x) = x - \frac{x^3}{6}$ So when we look at $\frac{\sin(x)-x}{x^3}$ near zero it is as if we have: $\frac{x - \frac{x^3}{6} - x}{x^3} = -\frac{1}{6}$. We can also say that near zero $\sin(x) - x$ goes to zero about 6 times faster then x^3.

(d) $\lim_{x\to 0^+} \frac{\int_0^x e^{-t^2}}{x} = \frac{0}{0}$ By L'Hopital's rule and applying the first fundamental theorem:

$$\lim_{x\to 0^+} \frac{\int_0^x e^{-t^2}}{x} = \lim_{x\to 0^+} \frac{e^{-x^2}}{1} = \lim_{x\to 0^+} \frac{1}{e^{x^2}} = 1$$

5. (a)

$$\lim_{n\to\infty} \frac{2n^2+3n+1}{n^2+2} = \lim_{x\to\infty} \frac{2x^2+3x+1}{n^2+2} = \lim_{x\to\infty} \frac{4x+3}{2x} = \lim_{x\to\infty} \frac{4}{2} = 2.$$

(b) Since for all n, $-1 \leq \sin n \leq 1$, we have

$$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

so by the squeeze theorem, we find $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(c) Letting $x = 1/n$ we have that $\lim_{n\to\infty} \left(1 + \frac{\pi}{n}\right)^n$ can be written as $\lim_{x\to 0}(1+\pi x)^{1/x}$ Taking the ln of the expression we may write $\lim_{x\to 0} \frac{ln(1+\pi x)}{x}$ Applying L'Hopital's rule we have $\lim_{x\to 0} \frac{\pi}{(1+\pi x)} =$ π Thus our limit was e^{π}. Also one can do this limit using ln but leaving n as it is.