Mathematics 1220 EXAM I Spring 2013

ANSWERS

1. a. By chain rule, $\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$.

   b. The derivative of $\tan^{-1} x$ is $\frac{1}{1+x^2}$, so the derivative of $\tan^{-1}(e^x)$ is $\frac{e^x}{1+e^{2x}}$, by the chain rule.

   c. Recall that $\lim_{x\to 0}(1+x)^{1/x} = e$. Therefore, setting $x = 1/n$, we have

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{x\to 0}(1+2x)^{1/x} = \left[\lim_{2x\to 0}(1+2x)^{1/2x}\right]^2 = e^2.$$

2. a. Let $u = e^x$, then $du = e^x dx$ and so

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2} = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C.$$

   b. Let $u = 3^x$, then $du = \ln(3)3^x dx$ and so

$$\int 3^x \cosh 3^x\,dx = \int \frac{\cosh u}{\ln 3}\,du = \frac{\sinh u}{\ln 3} + C = \frac{\sinh 3^x}{\ln 3} + C.$$

   c. $\int_0^1 xe^{-x^2}\,dx$ Let $u = -x^2$ then $du = -2xdx$ and our integral becomes.

$$\frac{-1}{2}\int_0^{-1} e^u\,du = \frac{1}{2}\int_{-1}^{0} e^u\,du = \frac{1}{2}(e^0 - e^{-1}) = \frac{1}{2} - \frac{1}{2e}$$

   d.

$$\int_1^{e^{125}} \frac{dt}{t} = \ln(e^{125}) - \ln(1) = 125$$

3. We are told that an initial bacteria population $P_0 = 100$ doubles every 30 min $=$ 0.5 hours. So the population model is $P(t) = P_0 2^{2t}$, with t in hours. Therefore $P(3) = (100)2^{2(3)} = (100)2^6 = 6400$.

4. The rate of change $\frac{dP(x)}{dx}$ of the pressure $P(x)$ is proportional to the pressure, i.e., $\frac{dP(x)}{dx} = kP(x)$, where k is the constant of proportionality. This differential equation has the solution $P(x) = P_0\,e^{kx}$, where $P_0 = P(0)$. The pressure at 6000 meters is half its value P_0 at sea level, i.e., $P_0/2 = P(6000) = P_0\,e^{6000k}$. Therefore $k = -\ln(2)/6000$. We are assuming that we know the pressure P_0 at sea level.

5. Our integrating factor is $\exp(\int -3\,dx) = e^{-3x}$, so that $\frac{d}{dx}\left(ye^{-3x}\right) = x$. Integrating both sides of this equation gives $ye^{-3x} = \frac{x^2}{2} + C$. Multiplying both sides of this equation by e^{3x} yields $y = \frac{x^2}{2}e^{3x} + Ce^{3x}$. The initial condition $y(0) = 4$ provides the value of C: $4 = y(0) = \frac{0^2}{2}e^0 + Ce^0 = C$. Therefore, $y(x) = \frac{x^2}{2}e^{3x} + 4e^{3x}$.

6. We are given $k = 0.5$, $T = 40$ and $\theta(0) = 60$ (assuming that $t = 0$ at 6am). After separating variables and integrating we have $\int \frac{d\theta}{\theta - T} = -\int k dt$. Therefore, we have that $\ln(\theta - T) = -kt + C$. Solving for $\theta(t)$ gives $\theta(t) = T + C_1 e^{-kt}$, where $C_1 = e^C$. The initial condition $\theta(0) = 60$ provides the value of C_1: $60 = \theta(0) = T + C_1 e^0 = T + C_1$, i.e., $C_1 = 60 - T$. Therefore, our model becomes $\theta(t) = T + (60 - T) e^{-kt}$. We now want to find out the time of death t_d, when the corpse last had a body temperature of 98.6°F.

$$\begin{aligned} 98.6 = \theta(t_d) &= 40 + 20e^{-0.5t_d} \\ \rightarrow \quad t_d &= -2\ln\left(\frac{58.6}{20}\right) \approx -2.15 \approx -2\text{hrs } 9\text{min}. \end{aligned}$$

Therefore, the person died at 6am − 2hrs 9min = 3:51am.