Mathematics 1220 PRACTICE EXAM II Spring 2013

ANSWERS

1. (a) Let $x = n$ and note that $\sqrt[x]{x} = x^{1/x} = \exp(\ln x/x)$. Use L'Hôpital's rule to show that $\lim_{x\to\infty} \ln x/x = 0$, hence $\lim_{x\to\infty} x^{1/x} = 1$. Now use L'Hôpital's rule to show that $\lim_{x\to\infty} \dfrac{x^{1/x}-1}{1/x} = \lim_{x\to\infty} -x^{1/x}(\ln x - 1)$ diverges, $\longrightarrow +\infty$.

   (b) Let $y = (\cos x)^{\csc x}$, and apply L'Hôpital's rule to $\ln y$, giving $y \to 1$.

   (c) Apply L'Hôpital's rule 25 times to x^{25}/e^x, giving 0.

   (d) Apply L'Hôpital's rule: $\lim_{x\to\infty} \dfrac{xe^x}{e^{x^2/2}} = \lim_{x\to\infty} e^{(\frac{-x^2}{2}+x)}\left(1+\dfrac{1}{x}\right)$. Since $\dfrac{-x^2}{2}+x = \dfrac{x}{2}(2-x) < 0$ for $x > 2$, the limit is 0.

   (e) $\lim_{x\to-\infty}(e^{-x}-x) = \lim_{x\to+\infty}(e^x+x) \longrightarrow +\infty$.

   (f) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator, which gives $\sin 1$ as the limit.

   (g) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator. Another application of L'Hôpital's rule gives $1/3$.

   (h) Straight forward asymptotics show that the limit is 3

   (i) Apply L'Hôpital's rule twice to show that the limit is $-1/2$

2. (a) 0 for $m = n$ and $m \neq n$, use $\sin mx \cos nx = \dfrac{1}{2}[\sin(m+n)x + \sin(m-n)x]$.

   (b) π, use $\sin mx \sin nx = -\dfrac{1}{2}[\cos 2mx - 1]$ for $m = n$.

   (c) $\dfrac{u^2}{2} + C, \quad u = \ln(\cosh x)$.   (d) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{u+1}{2}\right) + C, \quad u = e^x$. Then complete the square.

   (e) $\displaystyle\int \frac{3x-1}{x^2-4}dx = \int\left(\frac{7}{4}\frac{1}{x+2} + \frac{5}{4}\frac{1}{x-2}\right)dx = \frac{7}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C$

   (f) $\dfrac{x}{2}[\cos(\ln x) + \sin(\ln x)]$, use two integration by parts, the first with $u = \cos(\ln x)$, $dv = dx$, and second with $u = \sin(\ln(x))$ $dv = dx$ to solve for the integral.

   (g) $\dfrac{1}{2}(A\ln|x| + B\ln|x+3|) + C, \quad \dfrac{1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}$, $A = 1/3, B = -1/3$.

   (h) $-\dfrac{e^{-u}}{2} + C$, $u = t^2 + 2t + 5$.

   (i) $-\dfrac{1}{3}\sin^2(x)\cos(x) - \dfrac{2}{3}\cos(x)$ use the identity $\sin^2 x + \cos^2 x = 1$.

   (j) $xe^x - e^x + C$, do integration by parts with $u = x$ and $dv = e^x dx$.

   (k) $A\ln|x+5| + B\ln|x-2| + C$, $\frac{3x-13}{x^2+3x-10} = \frac{A}{x+5} + \frac{B}{x-2}$, $A = 4$, $B = -1$.

3. Section 7.1

   9. $2(4+z^2)^{\frac{3}{2}} + C$. Use the substitution $u = 4 + z^2$.

   19. $-\frac{1}{2}\cos(\ln(4x^2)) + C$. Use the substitution $u = \ln 4x^2$.

34. $-\frac{1}{2}\cot 2x - \frac{1}{2}\csc 2x + C$. Use $D_x \cot x = -\csc^2 x$ and $D_x \csc x = -\csc x \cot x$

4. Section 7.2

17. $\frac{2}{9}\left(e^{\frac{3}{2}} + 2\right)$. Use $u = \ln t$ and $dv = \sqrt{t}\,dt$.

39. $z\ln^2 z - 2z\ln z + 2z + C$. First use $u = \ln^2 z$ and $dv = dz$. Then use $u = \ln z$ and $dv = dz$.

41. $\frac{1}{2}e^t(\sin t + \cos t) + C$. First use $u = e^t$ and $dv = \cos t\,dt$. Then use $u = e^t$ and $dv = \sin t\,dt$.

5. Section 7.3

5. $\frac{8}{15}$. Write $\cos^5\theta = (1-\sin^2\theta)^2\cos\theta$, multiply out the quadratic term, and then use the substitution $u = \sin\theta$.

15. $\frac{1}{16}w - \frac{1}{32}\sin 2w - \frac{1}{24}\sin^3 w + C$. Use double angle formulas to write $\sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right) = \left(\frac{1-\cos w}{2}\right)^2\left(\frac{1+\cos w}{2}\right)$ and multiply everything out. Then use the cosine double angle formula again and write $\cos^3 w = (1-\sin^2 w)\cos w$.

30. 0. Use the identity $\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$.

6. (a) Let $u = \ln x$, then $p = 1/2$ for infinite domain $\Rightarrow$ divergence.
   (b) Bound the integrand above with C/x^p, for any p such that $1 < p < 3/2$, like $p = 5/4 \Rightarrow$ convergence.
   (c) $|e^{-x}\cos x| \le e^{-x} \Rightarrow$ convergence.
   (d) Near $x = 0$, $(e^{-x^2}/x^2) \sim (1/x^2) \Rightarrow$ divergence.
   (e) 16,000 integration by parts reduces the integral to a purely decaying exponential $\Rightarrow$ convergence; or use comparison $x^{16{,}000}e^{-x} < e^{-x/2}$ for sufficiently large x, but you must show this!
   (f) Near $x = 0$, integrand $\sim 1/x^{2/3} \Rightarrow$ convergence.

7. (a) diverges, as $\lim_{2n\to\infty}(-1)^{2n}\frac{2n}{2n+2} = 1$, but $\lim_{2n+1\to\infty}(-1)^{2n+1}\frac{2n+1}{(2n+1)+2} = -1$.
   (b) converges to $\frac{1}{2}$ as $\frac{\sqrt{n^2+4}}{2n+1} \sim \frac{n}{2n} = \frac{1}{2}$.
   (c) converges to 0. Apply the squeeze theorem to $\frac{-1}{n^{1/2}} \le \frac{\cos(2n)}{n^{1/2}} \le \frac{1}{n^{1/2}}$.
   (d) converges to 0 as $\ln\left(\frac{n}{n+1}\right) \sim \ln\left(\frac{n}{n}\right) = 0$.