Mathematics 2210–2,4 **Fall 2013**

PRACTICE EXAM I SOLUTIONS

1. Consider the vectors $\vec{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\vec{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

   (a) Find the length of $\vec{u}$.

   (b) Find $\vec{N} = \vec{u} \times \vec{v}$.

   (c) Find the cartesian equation of the plane with normal $\vec{N}$ through the point $P_0 = (1, 0, -1)$.

   (d) Find the vector projection of $\vec{v}$ onto $\vec{u}$.

   SOLUTION.

   a) $||\vec{u}|| = \sqrt{1+4+1} = \sqrt{6}$.

   b) $\vec{u} \times \vec{v} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} - (-6\mathbf{k} - 4\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

   c) $2x + 2y + 2z = C$
   $2(1) + 0 + 2(-1) = C = 0$
   $2x + 2y + 2z = 0$

   d) $\frac{\vec{v}\cdot\vec{u}}{||u||^2}\vec{u} =$

   $\frac{3+8+1}{1+4+1}(1, -2, 1) = (2, -4, 2)$

2. Determine the area of the triangle with vertices $P = (0, 0), Q = (3, 2), R = (1, 4)$.

   SOLUTION.

   Use $PQ =< 3, 2 >$ as the base of the triangle, with length $\sqrt{13}$. Find the height of the triangle by finding the length of the orthogonal projection of $PR =< 1, 4 >$ onto PQ. First find the vector projection:

   $$\frac{(1, 4) \cdot (3, 2)}{||(3, 2)||} < 3, 2 >= \frac{3+8}{4+9} < 3, 2 >$$
   $$=< 33/13, 22/13 >$$

   The orthogonal projection is $< 1, 4 >$ minus this vector:

   $$< 1, 4 > - < 33/13, 22/13 >=< -20/13, 30/13 >$$
   $$|| < -20/13, 30/13 > || = \sqrt{400/169 + 900/169} = \sqrt{1300/169} = 10/13\sqrt{13}$$

   Thus the triangle's area is $\frac{1}{2}\sqrt{13}\frac{10}{13}\sqrt{13} = 5$.

3. Rewrite the vector $\vec{v} = 4\mathbf{i} + 5\mathbf{j}$, in the orthonormal basis $e_1 = (\sqrt{3}/2, 1/2), e_2 = (-1/2, \sqrt{3}/2)$. In other words, expand or write $\vec{v}$ as $\vec{v} = ae_1 + be_2$ where a and b are scalar values.

SOLUTION.

The scalars a and b are determined by projecting $\vec{v}$ onto each of the basis vectors e_1 and e_2. Note that both e_1 and e_2 are unit vectors, so have length 1. This simplifies the vector projection formula:

$$a = \frac{\vec{v} \cdot e_1}{||e_1||^2} = \vec{v} \cdot e_1 = (4, 5) \cdot (\sqrt{3}/2, 1/2) = 2\sqrt{3} + 5/2$$

$$b = \vec{v} \cdot e_2 = (4, 5) \cdot (-1/2, \sqrt{3}/2) = -2 + \frac{5\sqrt{3}}{2}$$

4. Find the work done by the force $\vec{F} = 6\mathbf{i} + 8\mathbf{j}$ pounds in moving an object from $(1, 0)$ to $(6, 8)$ where distance is in feet.

SOLUTION.

The formula for work done is Work = **Force** x **Distance**. The distance vector is from (1, 0) to (6, 8), or a total of $< 5, 8 >$.

Work $= (6, 8) \cdot (5, 8) = 30 + 64 = 94$ lb-ft.

5. Given three points: $A = (0, 5, 3), B = (2, 7, 0), C = (-5, -3, 7)$

   (a) Which point is closest to the xz-plane? Explain your reasoning.

   (b) Which point lies on the xy-plane? Explain your reasoning.

SOLUTION.

a) The distance of a point from the xz-plane is simply the absolute value of the y-coordinate. You should try to draw a picture to visualize this. Thus, C is the closest with a y-value of -3.

b) A point is on the xy-plane if its z-coordinate is 0. Thus, point B is the point we're looking for.

6. Determine the equation of the plane spanned by the vectors:

$$\vec{u} = 1\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\vec{v} = 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$$

and which contains the origin.

SOLUTION.

Since both $\vec{u}$ and $\vec{v}$ lie in the plane and are in different directions, we may take their cross product to find the Normal vector to the plane and determine its equation:

$$\vec{u} \times \vec{v} = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) - (6\mathbf{k} - 12\mathbf{i} + 4\mathbf{j}) = 24\mathbf{i} - 8\mathbf{j}$$

Thus the equation for the plane is $24x - 8y = C$. The plane contains the origin, so $24(0) - 8(0) = 0 = C$. So the solution is $24x - 8y = 0$.

7. Find the curvature of the line parameterized by $\vec{r}(t) = (1, 1, 1) + (2, 3, 4)t$.

SOLUTION.

$r(t) = (1 + 2t, 1 + 3t, 1 + 4t)$
$r'(t) = (2, 3, 4)$
$T(t) = r'(t)/\sqrt{4 + 9 + 16} = 1/\sqrt{29}(2, 3, 4)$
$T'(t) = 0$
$\kappa = ||T'(t)||/||r'(t)|| = 0/\sqrt{29} = 0$

NOTE: Straight lines always have 0 curvature!

8. Find the arc length of the helix

$$\vec{r}(t) = a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} + ct\mathbf{k}$$

for $0 \leq t \leq 2\pi$.

SOLUTION.

$r'(t) = a\cos(t)\mathbf{i} - a\sin(t)\mathbf{j} + c\mathbf{k}$

Arc Length $= \int_0^{2\pi} ||r'(t)||dt = \int_0^{2\pi} \sqrt{a^2cos^2(t) + a^2sin^2(t) + c^2}dt$

$= \int_0^{2\pi} \sqrt{a^2 + c^2}dt = 2\pi\sqrt{a^2 + c^2}$

9. Find the equation of the plane orthogonal to the curve

$$\vec{r}(t) = (8t^2 - 4t + 3)\mathbf{i} + (\sin(t) - 4t)\mathbf{j} - \cos(t)\mathbf{k}$$

at the point $t = \pi/3$.

SOLUTION.

The Normal vector for the plane will be the tangent vector to the curve at $t = \pi/3$. We will also need to know $r(\pi/3)$.

$$r(\pi/3) = (8\pi^2/9 - 4\pi/3 + 3, \sqrt{3}/2 - 4\pi/3, -1/2)$$
$$r'(t) = (16t - 4)\mathbf{i} + (\cos(t) - 4)\mathbf{j} + \sin(t)\mathbf{k}$$
$$r'(\pi/3) = (16\pi/3 - 4, 1/2 - 4, \sqrt{3}/2)$$

Thus the equation for the plane is

$$(16\pi/3 - 4)x + (-7/2)y + (\sqrt{3}/2)z = C$$

Using the point from above,

$$(16\pi/3 - 4)(8\pi^2/9 - 4\pi/3 + 3) + (-7/2)(\sqrt{3}/2 - 4\pi/3) + (\sqrt{3}/2)(-1/2) = C$$

So the equation for the plane is
$(16\pi/3 - 4)x + (-7/2)y + (\sqrt{3}/2)z =$
$(16\pi/3 - 4)(8\pi^2/9 - 4\pi/3 + 3) + (-7/2)(\sqrt{3}/2 - 4\pi/3) + (\sqrt{3}/2)(-1/2)$

10. Determine the curvature κ of the helical curve parametrized by:

$$\vec{r}(t) = 7\sin(3t)\mathbf{i} + 7\cos(3t)\mathbf{j} + 14t\mathbf{k}$$

at $t = \pi/9$.

SOLUTION.

$r'(t) = 21\cos(3t)\mathbf{i} - 21\sin(3t)\mathbf{j} + 14\mathbf{k}$

$||r'(t)|| = \sqrt{21^2\cos^2(3t) + 21^2\sin^2(3t) + 14^2} = \sqrt{441 + 196} = \sqrt{637}$

$T(t) = 1/\sqrt{637}(21\cos(3t)\mathbf{i} - 21\sin(3t)\mathbf{j} + 14)$

$T'(t) = 1/\sqrt{637}(-63\sin(3t)\mathbf{i} - 63\cos(3t)\mathbf{j})$

$||T'(t)|| = 1/\sqrt{637}(\sqrt{63^2\sin^2(3t) + 63^2\cos^2(3t)}) = 63/\sqrt{637}$

$\kappa = \frac{||T'(t)||}{||r'(t)||} = \frac{63}{637}$

Note that the curvature is constant for all t, in particular for $t = \pi/9$.

11. Given that the acceleration of a particle's motion is

$$\vec{a}(t) = -9\cos(3t)\mathbf{i} + -9\sin(3t)\mathbf{j} + 2t\mathbf{k}.$$

And the particle has initial velocity $\vec{v}_0 = \mathbf{i} + \mathbf{k}$ and intial position $\vec{x}_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(a) Determine the velocity function $\vec{v}(t)$.

(b) Determine the position function $\vec{x}(t)$.

SOLUTION.

$$\vec{v}(t) = -3\sin(3t)\mathbf{i} + 3\cos(3t)\mathbf{j} + t^2\mathbf{k} + \vec{C}$$
$$\vec{v}(0) = \mathbf{i} + \mathbf{k} = (0, 3, 0) + \vec{C}$$
$$\vec{C} = (1, -3, 1)$$
$$\vec{v}(t) = (-3\sin(3t) + 1)\mathbf{i} + (3\cos(3t) - 3)\mathbf{j} + (t^2 + 1)\mathbf{k}$$
$$\vec{x}(t) = (cos(3t) + t)\mathbf{i} + (sin(3t) - 3t)\mathbf{j} + (t^3/3 + t)\mathbf{k} + \vec{D}$$
$$\vec{x}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 0, 0) + \vec{D}$$
$$\vec{D} = (0, 1, 1)$$
$$\vec{x}(t) = (cos(3t) + t)\mathbf{i} + (sin(3t) - 3t + 1)\mathbf{j} + (t^3/3 + t + 1)\mathbf{k}$$

12. Let $\vec{u} = (2, 2)$ and $\vec{v} = (3, -1)$. Find $\vec{u} + \vec{v}$ and illustrate this vector addition with a diagram in the plane, showing $\vec{u}$, $\vec{v}$ and the resultant vector.

SOLUTION.

$\vec{u} + \vec{v} = (2, 2) + (3, -1) = 5, 1)$

Note that the resultant vector points from the origin to the head of $\vec{v}$, after translating $\vec{v}$ so that its tail is located at the head of $\vec{u}$.