Mathematics 1220 PRACTICE EXAM III Spring 2017

ANSWERS

1. For $\ln(1+x)$, substitute in $-x$ for x in the geometric series and integrate term by term to see that,
   $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + ...$, radius $= 1$.
   For $1/(1+x)^2$, substitute in $-x$ for x in the geometric series, take the derivative term by term, and multiply by -1 to see that,
   $1/(1+x)^2 = 1 - 2x + 3x^2 - 4x^3 + ...$, radius $= 1$.

2. $\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\left(1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 + \ldots\right)$, radius $= 2$

3. $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$,
   $2e^{i\pi/2} = 2\cos(\pi/2) + 2i\sin(\pi/2) = 0 + 2(1)i = 2i$,
   $2e^{-i3\pi/2} = 2\cos(-3\pi/2) + 2i\sin(-3\pi/2) = 0 + 2(1)i = 2i$,
   $\pi e^{i2\pi} = \pi\cos(2\pi) + \pi i\sin(2\pi) = \pi 1 + 0i = \pi$

4. $\cosh(x) = \frac{e^x + e^{-x}}{2}$
   $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$
   and $e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! + \cdots$.
   Plugging these into the definition of $\cosh(x)$ and simplifying we obtain:
   $\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \cdots$. The radius of convergence is infinite since that is the case for the series for e^x

5. Apply the ratio test and set the result to be less than 1, which would imply convergence, and solve for the x values that fit that condition. Next plug in the endpoints of the interval you find into the series and determine convergence at those points. Note that for part b, $\lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right| = \phi = (1+\sqrt{5})/2$, ϕ is the golden ratio!

   (a) $(-1, 1/3)$, conditionally convergent at -1, divergent at $1/3$
   (b) $(-1/\phi, 1/\phi)$, $\phi = (1+\sqrt{5})/2$

6. (a) $-e^{-n} \le e^{-n}\sin n \le e^{-n}$ and $\lim_{n\to\infty} e^{-n} = 0$. Thus, $\lim_{n\to\infty} e^{-n}\sin(n) = 0$ by the squeeze theorem.

   (b) Let $L = \lim_{n\to\infty}(2n)^{1/2n}$ so that, $\ln(L) = \lim_{n\to\infty}(1/2n)\ln(2n)$. Apply L'Hospital's rule to see that $\ln(L) = 0$ thus $L = 1$

   (c) $\lim_{n\to\infty}(1-1/n)\cos(n\pi) = \lim_{n\to\infty}(1-1/n)\lim_{n\to\infty}\cos(n\pi) = 1\lim_{n\to\infty}\cos(n\pi)$, however $\lim_{n\to\infty}\cos(n\pi)$ does not exist due to oscillation.

   (d) Note that,

   $$\lim_{n\to\infty}\sum_{k=1}^{\infty} exp\left(\frac{k}{n}\right)^2 \frac{1}{n}$$

   is just the limit of the Riemann sum for $\int_0^1 e^{x^2}\, dx$. This is an integral that is non-trival. To find the value we use the Taylor Series for $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + x^8/4! + \cdots$ and integrate it term by term, then we can plug in our bounds to find that, $\int_0^1 e^{x^2}\, dx = 1 + \frac{1}{3} + \frac{1}{5\cdot 2!} + \frac{1}{7\cdot 3!} + ...$

7. The sum of the distances of each jump is $1 + 3/4 + (3/4)^2 + (3/4)^3 + \cdots$ This is just a geometric series with $r = 3/4$ thus, $1 + 3/4 + (3/4)^2 + (3/4)^3 + \cdots = \frac{1}{1-3/4} = 4meters$

8. (a) Diverges in absolute value, but converges conditionally (Alternating Series Test)

   (b) $\sqrt{1 - \cos\left(\frac{1}{n}\right)} \sim \frac{1}{n} \Longrightarrow$ divergence

   (c) converges absolutely (ratio test)

   (d) diverges (n^{th} term test)

   (e) converges absolutely (integral test)

   (f) converges absolutely (ratio test)

9. $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$, $\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots$, and we have $\frac{d^2y}{dx^2} = -y$ and the initial conditions $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$, so we find

$$a_2 = \frac{-a_0}{2} = 0 \quad a_3 = \frac{-a_1}{6} = \frac{-1}{3!}$$
$$a_4 = \frac{-a_2}{12} = 0 \quad a_5 = \frac{-a_3}{20} = \frac{1}{5!}$$

and so on, so $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sin x$.

10. Finding the first four terms. $f(x) = \sqrt{x-2}$, $f'(x) = \frac{1}{2\sqrt{x-2}}$, $f''(x) = \frac{-1}{4(x-2)^{3/2}}$ $f'''(x) = \frac{3}{8(x-2)^{5/2}}$ Thus $f(3) = 1, f'(3) = \frac{1}{2}, f''(3) = -\frac{1}{4}, f'''(3) = \frac{3}{8}$. Using taylor's theroem

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$ With $f(x) = \sqrt{x-2}$ and $a = 3$ we get
$\sqrt{x-2} = 1 + \frac{1}{2}(x-3) - \frac{1}{8}(x-3)^2 + \frac{3}{48}(x-3)^3 + \ldots$
The radius of convergence is $[2, 4]$ since we require $|x-3| < 1$ and we have an alternating series with limit 0 as $n \to \infty$ at each end point on $[2, 4]$