COMPLETENESS OF THE EIGENFUNCTIONS FOR GRIFFITH CRACKS IN PLATES OF FINITE THICKNESS

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Introduction.

E.S. Folias [1] has constructed the displacement and stress fields near a Griffith crack as an expansion in eigenfunctions. The eigenfunctions were derived by an operational method due to Lurê [2] and the question of completeness arises. The purpose of this report is to prove the completeness by a constructive method. The method employed is to solve the boundary value problem by Fourier analysis and to evaluate the resulting integrals as residue series. The terms in these series are precisely the eigenfunctions used by Folias.

Notation.

A system of Cartesian coordinates (x,y,z) is used. The plate occupies the region defined by

$$-\infty < x < \infty \ , \quad -\infty < y < \infty \ , \quad -h \leqslant z \leqslant h$$

The crack is defined by

$$-c \leqslant x \leqslant c \ , \quad y = 0 \ , \quad -h \leqslant z \leqslant h$$

The components of the displacement field in the stressed plate are

$$u(x,y,z) \;,\quad v(x,y,z) \;,\quad w(x,y,z)$$

The corresponding stress tensor components are

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2G\,\frac{\partial u}{\partial x}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_{zx}$$

$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2G\,\frac{\partial v}{\partial y}$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

$$\sigma_z = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2G\,\frac{\partial w}{\partial z}$$

The displacement field satisfies the field equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + a^2\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + a^2\,\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + a^2\,\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

where $a^2 = \frac{\lambda+G}{G}$. The boundary conditions are

$$\tau_{xz} = 0\ ,\quad \tau_{yz} = 0\ ,\quad \sigma_z = 0 \quad \text{for} \quad -\infty < x,y < \infty\ ,\quad z = \pm h$$

$$\tau_{xy} = 0\ ,\quad \tau_{zy} = 0\ ,\quad \sigma_y = -\sigma_0 \quad \text{for} \quad -c < x < c\ ,\quad y = 0\pm\ ,\quad |z| \leqslant h$$

$$u,v,w = O(1) \quad \text{for} \quad x^2 + y^2 \to \infty\ ,\quad |z| \leqslant h$$

Symmetries.

	x	y	z
u(x,y,z)	odd	even	even
v(x,y,z)	even	odd	even
w(x,y,z)	even	even	odd

If follows from the symmetries that

$$v = 0\ ,\quad \frac{\partial v}{\partial x} = 0\ ,\quad \frac{\partial v}{\partial z} = 0 \quad \text{for} \quad |x| > c\ ,\quad y = 0\ ,\quad |z| \leqslant h$$

$$\frac{\partial u}{\partial y} = 0\ ,\quad \frac{\partial w}{\partial y} = 0 \qquad \text{for} \quad |x| > c\ ,\quad y = 0\ ,\quad |z| \leqslant h$$

whence

$$\tau_{xy} = 0\ ,\quad \tau_{zy} = 0 \qquad \text{for} \quad |x| > c\ ,\quad y = 0\ ,\quad |z| \leqslant h$$

Moreover, if

$$\left.\begin{aligned} u^{\pm}(x,0,z) &= \lim_{\varepsilon\to 0} u(x,\pm\varepsilon,z) \\ [u](x,0,z) &= u^{+}(x,0,z) - u^{-}(x,0,z) \end{aligned}\right\} \text{etc.}$$

then

$$\left.\begin{aligned} [u] &= [\frac{\partial u}{\partial x}] = [\frac{\partial u}{\partial z}] = 0 \quad \text{for} \quad -\infty < x < \infty, \quad |z| \leq h \\ [w] &= [\frac{\partial w}{\partial x}] = [\frac{\partial w}{\partial z}] = 0 \quad \text{for} \quad -\infty < x < \infty, \quad |z| \leq h \\ [\frac{\partial v}{\partial y}] &= 0 \qquad\qquad \text{for} \quad -\infty < x < \infty, \quad |z| \leq h \end{aligned}\right\} \text{even in } y$$

while

$$[\frac{\partial u}{\partial y}] = 2(\frac{\partial u}{\partial y})^{+}, \quad [v] = 2v^{+}, \quad \text{etc.}\} \text{ odd in } y$$

Note: these vanish at points of continuity.

Application of the Fourier Transform in x and y.

Define

$$\hat{u}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ipx} u(x,y,z)dx$$

etc.

$$\tilde{u}(p,q,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-iqy} \hat{u}(p,y,z)dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(px+qy)} u(x,y,z)dxdy$$

etc.

Then

$$\left(\frac{\partial u}{\partial x}\right)^{\wedge} = ip\hat{u}\ , \quad \left(\frac{\partial^2 u}{\partial x^2}\right)^{\wedge} = -p^2\hat{u}$$

$$\left(\frac{\partial u}{\partial y}\right)^{\wedge} = \frac{\partial \hat{u}}{\partial y}\ , \quad \text{etc.}$$

However, u,v,w and their derivatives may have discontinuities across the crack. Note that, if $f^{(k)}(y) \in L_1(\mathbb{R})$ for $k = 0,1,2$ and $f \in C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-)$ and $f(0\pm)$, $f'(0\pm)$ are finite

$$(f')^{\wedge}(q) = \frac{1}{(2\pi)^{1/2}} \left(\int_0^{\infty} + \int_{-\infty}^{0} \right) e^{-iqy}\ f'(y)dy$$

$$= \frac{1}{(2\pi)^{1/2}} \left(e^{-iqy}\ f(y)\Big]_{0+}^{\infty} + e^{-iqy}\ f(y)\Big]_{-\infty}^{0-} + iq \int_{-\infty}^{\infty} e^{-iqy}\ f(y)dy \right)$$

$$= -\frac{[f]}{(2\pi)^{1/2}} + iq\ \hat{f}(q)$$

and hence

$$(f'')^{\wedge}(q) = -\frac{[f']}{(2\pi)^{1/2}} + iq\ (f')^{\wedge}(q)$$

$$= -\frac{[f']}{(2\pi)^{1/2}} - iq\ \frac{[f]}{(2\pi)^{1/2}} - q^2\ \hat{f}(q)$$

These results and the symmetries (p. 3) imply

$$\left(\frac{\partial u}{\partial y}\right)^{\sim} = iq\,\tilde{u}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)^{\sim} = -\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\partial u}{\partial y}\right)^{\wedge}_0 - q^2\,\tilde{u} \qquad \left(\left(\frac{\partial u}{\partial y}\right)^{\wedge}_0 = \left(\frac{\partial u}{\partial y}\right)^{\wedge}(p,0+,z)\right)$$

etc.

$$\left(\frac{\partial v}{\partial y}\right)^{\sim} = -\left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0 + iq\,\tilde{v}$$

$$\left(\frac{\partial^2 v}{\partial y^2}\right)^{\sim} = -iq\left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0 - q^2\,\tilde{v}$$

$$\left(\frac{\partial w}{\partial y}\right)^{\sim} = iq\,\tilde{w}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^{\sim} = -\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\partial w}{\partial y}\right)^{\wedge}_0 - q^2\,\tilde{w}$$

Taking the Fourier transform of the field equations (p. 2) and using the above results gives

$$\frac{d^2\tilde{u}}{dz^2} - p^2\,\tilde{u} - q^2\,\tilde{u} - \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\partial u}{\partial y}\right)^{\wedge}_0 + a^2\left(-p^2\,\tilde{u} - pq\,\tilde{v} - ip\left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0 + ip\,\frac{d\tilde{w}}{dz}\right) = 0$$

$$\frac{d^2\tilde{v}}{dz^2} - p^2\,\tilde{v} - q^2\,\tilde{v} - iq\left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0 + a^2\left(-pq\,\tilde{u} - q^2\,\tilde{v} - iq\left(\frac{2}{\pi}\right)^{1/2}\tilde{v}_0 + iq\,\frac{d\tilde{w}}{dz}\right) = 0$$

$$\frac{d^2\tilde{w}}{dz^2} - p^2\,\tilde{w} - q^2\,\tilde{w} - \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\partial w}{\partial y}\right)^{\wedge}_0 + a^2\left(ip\,\frac{d\tilde{u}}{dz} + iq\,\frac{d\tilde{v}}{dz} - \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\partial v}{\partial z}\right)^{1}_0 + \frac{d^2\tilde{w}}{dz^2}\right) = 0$$

or

$$\frac{d^2\tilde{u}}{dz^2} - (p^2 + q^2)\tilde{u} - a^2 p(p\tilde{u} + q\tilde{v} - i\frac{d\tilde{w}}{dz}) = (\frac{2}{\pi})^{1/2}[(\frac{\partial u}{\partial y})^{\wedge}_0 + ip\, a^2\, \tilde{v}_0]$$

$$\frac{d^2\tilde{v}}{dz^2} - (p^2 + q^2)\tilde{v} - a^2 q(p\tilde{u} + q\tilde{v} - i\frac{d\tilde{w}}{dz}) = (\frac{2}{\pi})^{1/2}[iq\, \tilde{v}_o + iq\, a^2\, \tilde{v}_0]$$

$$(1+a^2)\frac{d^2\tilde{w}}{dz^2} - (p^2+q^2)\tilde{w} + i\, a^2(p\frac{d\tilde{u}}{dz} + q\frac{d\tilde{v}}{dz}) = (\frac{2}{\pi})^{1/2}[(\frac{\partial w}{\partial y})^{\wedge}_0 + a^2(\frac{\partial v}{\partial z})^{\wedge}_0]$$

Note that

$$f(x) \text{ is even } \iff \hat{f}(p) \text{ is even and}$$

$$\hat{f}(p) = (\frac{2}{\pi})^{1/2}\int_0^\infty \cos px\, f(x)dz = F_c f(p)$$

$$f(x) = (\frac{2}{\pi})^{1/2}\int_0^\infty \cos px\, \hat{f}(p)dp$$

while

$$f(x) \text{ is odd } \iff \hat{f}(p) \text{ is odd and}$$

$$\hat{f}(p) = -i\,(\frac{2}{\pi})^{1/2}\int_0^\infty \sin px\, f(x)dz = -i\, F_s f(p)$$

$$f(x) = i\,(\frac{2}{\pi})^{1/2}\int_0^\infty \sin px\, \hat{f}(p)dp = (\frac{2}{\pi})^{1/2}\int_0^\infty \sin px\, F_s f(p)dp$$

The analogous formulas are valid for functions of y . Hence the symmetries, p. 3, imply

$$\tilde{u} = -i\,U\,, \quad \tilde{v} = -i\,V\,, \quad \tilde{w} = W$$

where U, V, W are real-valued. In fact,

$$U = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin px \cos qy \; u(x,y,z)\,dxdy$$

$$V = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos px \sin qy \; v(x,y,z)\,dxdy$$

$$W = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos px \cos qy \; w(x,y,z)\,dxdy$$

Hence the differential equations for $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ on p. 7 are equivalent to

$$\frac{d^2U}{dz^2} - a^2p\,\frac{dW}{dz} - a^2p(p\,U + q\,V) - (p^2 + q^2)U = \left(\frac{2}{\pi}\right)^{1/2}\left[F_s\left(\frac{\partial u}{\partial y}\right)_0 - pa^2F_c v_0\right]$$

$$\frac{d^2V}{dz^2} - a^2q\,\frac{dW}{dz} - a^2q(p\,U + q\,V) - (p^2 + q^2)V = \left(\frac{2}{\pi}\right)^{1/2}\left[\, - (1+a^2)qF_c v_0\right]$$

$$(1+a^2)\,\frac{d^2W}{dz^2} + a^2\left(p\,\frac{dU}{dz} + q\,\frac{dV}{dz}\right) - (p^2 + q^2)W = \left(\frac{2}{\pi}\right)^{1/2}\left[F_c\left(\frac{\partial w}{\partial y}\right)_0 + a^2F_c\left(\frac{\partial v}{\partial z}\right)_0\right]$$

This can be written as a 2nd order 3×3 matrix system of ODE's, namely

$$L\begin{pmatrix}U\\V\\W\end{pmatrix} - (p^2 + q^2)\begin{pmatrix}U\\V\\W\end{pmatrix} = \begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix}$$

where, if $\bar{U} = (U,V,W)^T$ (T = transpose)

$$L\bar{U} = A\frac{d^2\bar{U}}{dz^2} + B\frac{d\bar{U}}{dz} + C\bar{U}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & -a^2p \\ 0 & 0 & -a^2q \\ a^2p & a^2q & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -a^2p^2 & -a^2pq & 0 \\ -a^2pq & -a^2q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that

$$A^T = A\ , \quad B^T = -B\ , \quad C^T = C$$

It follows that L is formally selfadjoint with respect to the scalar product

$$(\bar{U},\bar{V}) = \int_{z_1}^{z_2} \bar{U}^T\ \bar{V}\ dz$$

In fact, integration by parts gives

$$(\overline{U},L\overline{V}) = \int_{z_1}^{z_2} (\overline{U}^T A \overline{V}'' + \overline{U}^T B \overline{V}' + \overline{U}^T C \overline{V})dz$$

$$= \overline{U}^T A \overline{V}' + \overline{U}^T B \overline{V}\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} (\overline{U}'^T A \overline{V}' + \overline{U}'^T B \overline{V} - \overline{U}^T C \overline{V})dz$$

$$= \overline{U}^T A \overline{V}' - \overline{U}'^T A \overline{V} + \overline{U}^T B \overline{V}\Big]_{z_1}^{z_2}$$

$$+ \int_{z_1}^{z_2} (\overline{U}''^T A - \overline{U}'^T B + \overline{U}^T C) \overline{V} dz$$

$$= [\overline{U},\overline{V}]_{z_1}^{z_2} + \int_{z_1}^{z_2} (A\overline{U}'' + B \overline{U}' + C \overline{U})^T \overline{V} dz$$

$$= [\overline{U},\overline{V}]_{z_1}^{z_2} + (L \overline{U}, \overline{V})$$

where

$$[\overline{U},\overline{V}] = \overline{U}^T A \overline{V}' - \overline{U}'^T A \overline{V} + \overline{U}^T B \overline{V}$$

If the index notation

$$\overline{U} = (U_1,U_2,U_3)^T , \quad \overline{V} = (V_1,V_2,V_3)^T$$

is used the bilinear form $[\overline{U},\overline{V}]$ can be written

$$[\bar{U},\bar{V}] = U_1V_1' + U_2V_2' + (1+a^2)U_3V_3'$$
$$- U_1'V_1 - U_2'V_2 - (1+a^2)U_3'V_3$$
$$- a^2pU_1V_3 - a^2qU_2V_3 + a^2pU_3V_1 + a^2qU_3V_2$$

Boundary Conditions Associated with L.

The symmetry properties of the displacement field wrt z (p. 3) imply that

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0 \ , \quad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0 \ , \quad (w)_{z=0} = 0$$

It follows that

$$\frac{dU(0)}{dz} = 0 \ , \quad \frac{dV(0)}{dz} = 0 \ , \quad W(0) = 0$$

Note that

B.C.1 $\qquad U_1'(0) = 0 \ , \quad U_2'(0) = 0 \ , \quad U_3(0) = 0$

is selfadjoint for L ; i.e.

$$\bar{U} \text{ and } \bar{V} \text{ satisfy B.C.1} \Rightarrow [\bar{U},\bar{V}]_{z=0} = 0$$

The B.C.'s at $z = \pm h$ imply corresponding B.C.'s for $\bar{U}$. To write then note that (p. 6)

$$\tilde{\tau}_{xz} = G(\frac{d\tilde{u}}{dz} + ip\ \tilde{w}) = -i\ G(\frac{dU}{dz} - p\ W)$$

$$\tilde{\tau}_{yz} = G(\frac{d\tilde{v}}{dz} + iq\ \tilde{w}) = -i\ G(\frac{dV}{dz} - q\ W)$$

$$\tilde{\sigma}_z = \lambda(ip\ \tilde{u} + iq\ \tilde{v} - (\frac{2}{\pi})^{1/2}\ \hat{v}_0 + \frac{d\tilde{w}}{dz}) + 2G\ \frac{d\tilde{w}}{dz}$$

$$= \lambda(p\ U + q\ V) + (\lambda + 2G)\ \frac{dW}{dz} - (\frac{2}{\pi})^{1/2}\ \lambda\ \hat{v}_0$$

$$= \lambda[p\ U + q\ V + \frac{a^2+1}{a^2-1}\ \frac{dW}{dz}] - (\frac{2}{\pi})^{1/2}\ \lambda\ \hat{v}_0$$

since

$$\lambda = \frac{2G}{m-2}\ ,\quad \frac{2G}{\lambda} = m-2\ ,\quad 1 + \frac{2G}{\lambda} = m-1 = \frac{a^2+1}{a^2-1}$$

$$a^2 = \frac{m}{m-2}\ ,\quad ma^2 - 2a^2 = m\ ,\quad m(a^2 - 1) = 2a^2$$

$$m = \frac{2a^2}{a^2-1}\ ,\quad \frac{\lambda}{2G} = \frac{a^2-1}{2}\ ,\quad \frac{m}{m-1} = \frac{2a^2}{a^2+1}$$

It follows that

$$\frac{dU(h)}{dz} - p\ W(h) = 0\ ,\quad \frac{dV(h)}{dz} - q\ W(h) = 0$$

$$p\ U(h) + q\ V(h) + \frac{a^2+1}{a^2-1}\ \frac{dW(h)}{dz} = (\frac{2}{\pi})^{1/2}\ \hat{v}_0(h)$$

Note that

B.C.2 $U_1'(h) - p\,U_3(h) = 0\ ,\quad U_2'(h) - q\,U_3(h) = 0$

$$p\,U_1(h) + q\,U_2(h) + \frac{a^2+1}{a^2-1}\,U_3'(h) = 0$$

is also selfadjoint for L. In fact, if $\overline{U}$ and $\overline{V}$ satisfy B.C.2 then

$$[\overline{U},\overline{V}]_{z=h} = p\,U_1V_3 + q\,U_2V_3 - (a^2-1)U_3\,(p\,V_1 + q\,V_2)$$

$$- p\,U_3V_1 - q\,U_3V_2 + (a^2-1)\,(p\,U_1 + q\,U_2)V_3$$

$$- a^2p\,U_1V_3 - a^2q\,U_2V_3 + a^2p\,U_3V_1 + a^2q\,U_3V_2 = 0$$

BV Problem for $\overline{U} = (U_1,U_2,U_3)^T = (U,V,W)^T$.

$$L\,\overline{U} - (p^2 + q^2)\overline{U} = \overline{F}\ ,\quad 0 < z < h$$

$$M_0\overline{U}(0) + N_0\overline{U}'(0) = 0$$

$$M_h\overline{U}(h) + N_h\overline{U}'(h) = \overline{G}(h)$$

where

$$\overline{F}(z) = (\tfrac{2}{\pi})^{1/2}\begin{bmatrix} F_s(\frac{\partial u}{\partial y})_0 - pa^2\,F_c\,v_0 \\ -\ (1+a^2)\,q\,F_c\,v_0 \\ F_c(\frac{\partial w}{\partial y})_0 + a^2F_c(\frac{\partial v}{\partial z})_0 \end{bmatrix}$$

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,\quad N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_h = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -q \\ p & q & 0 \end{pmatrix} , \quad N_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{a^2+1}{a^2-1} \end{pmatrix}$$

and

$$\overline{G}(h) = \begin{pmatrix} 0 \\ 0 \\ (\frac{2}{\pi})^{1/2} F_c \; v_0 \end{pmatrix}$$

<u>Method of Solving the BV Problem for U,V,W .</u>

To solve the BV problem the general solution of $L\,\overline{U} - (p^2 + q^2)\overline{U} = \overline{F}$ and BC at $z = 0$ will be constructed as a function of the parameters

$$u_0 = U(0) \;, \quad v_0 = V(0) \;, \quad w_0' = \frac{dW(0)}{dz}$$

The B.C.'s at $z = h$ will then be used to calculate u_0 , v_0 , w_0' .

<u>Solutions of the Equations $L\,\overline{U} - (p^2 + q^2)\overline{U} = \overline{0}$.</u>

This equation, written in terms of components $(U_1,U_2,U_3) = (U,V,W)$, is obtained from the system on p.8 by setting the right-hand side equal to 0 . Note that this system coincides with Luré, p. 150 (3.2.12) under the correspondence

$$-i\,U \Leftrightarrow u \;, \quad -i\,V \Leftrightarrow v \;, \quad W \Leftrightarrow w$$

$$ip \Leftrightarrow \partial_1 \;, \quad iq \Leftrightarrow \partial_2 \;, \quad -(p^2 + q^2) \Leftrightarrow D^2$$

$$ip(-i\,U) + iq(-i\,V) + \frac{dW}{dz} = p\,U + q\,V + \frac{dW}{dz} \Leftrightarrow \theta$$

Luré has given a complete solution of the system $L\,\bar{U} = (p^2 + q^2)\bar{U}$ in equation (3.2.15), (3.2.17). To adapt them to the present notation write

$$s^2 = p^2 + q^2 \ , \quad s = \sqrt{p^2 + q^2} \geq 0 \ , \quad D = is$$

$$\cos zD = \cos isz = \cosh sz \ , \quad \sin zD = \sin isz = i \sinh sz$$

Then the solution (3.2.15) becomes

$$U = (\cosh sz)u_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (p^2 u_0 + pqv_0 + pw_0')$$

$$V = (\cosh sz)v_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (pqu_0 + q^2 v_0 + qw_0')$$

$$W = \frac{\sinh sz}{s} w_0' + \frac{a^2}{2} \left(\frac{\sinh sz}{s} - z \cosh sz\right)(pu_0 + qv_0 + w_0')$$

This solution satisfies B.C.1 at $z = 0$. The solution (3.2.17) becomes

$$U = \frac{\sinh sz}{s} u_0' + \frac{a^2}{2(a^2+1)} \left(\frac{i \sinh sz}{-i\, s^3} + \frac{z \cosh sz}{+s^2}\right)(p^2 u_0' + pqv_0' + ps^2 w_0)$$

$$V = \frac{\sinh sz}{s} v_0' + \frac{a^2}{2(a^2+1)} \left(\frac{\sinh sz}{-s^3} + \frac{z \cosh sz}{s^2}\right)(pqu_0' + q^2 v_0' + qs^2 w_0)$$

$$W = (\cosh sz)w_0 - \frac{a^2}{2(a^2+1)} \frac{z \sinh sz}{s} (pu_0' + qv_0' + s^2 w_0)$$

This solution satisfies

<u>B.C.1'</u> $\qquad U(0) = 0 \ , \quad V(0) = 0 \ , \quad \dfrac{dW(0)}{dz} = 0$

and

$$u_0' = \frac{dU(0)}{dz} \ , \quad v_0' = \frac{dV(0)}{dz} \ , \quad w_0 = W(0)$$

Solution Basis for $L\,\overline{U} = (p^2 + q^2)\overline{U}$.

$u_0 = 1$, $v_0 = w_0' = 0$ gives

$$\begin{pmatrix} U^1 \\ V^1 \\ W^1 \end{pmatrix} = \begin{pmatrix} \cosh sz + \frac{a^2}{2} \frac{z \sinh sz}{s} p^2 \\ \frac{a^2}{2} \frac{z \sinh sz}{s} pq \\ \frac{a^2}{2} \left(\frac{\sinh sz}{s} - z \cosh sz \right) p \end{pmatrix}$$

$u_0 = 0$, $v_0 = 1$, $w_0' = 0$ gives

$$\begin{pmatrix} U^2 \\ V^2 \\ W^2 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2} \frac{z \sinh sz}{s} pq \\ \cosh sz + \frac{a^2}{2} \frac{z \sinh sz}{s} q^2 \\ \frac{a^2}{2} \left(\frac{\sinh sz}{s} - z \cosh sz \right) q \end{pmatrix}$$

$u_0 = v_0 = 0$, $w_0' = 1$ gives

$$\begin{pmatrix} U^3 \\ V^3 \\ W^3 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2} \frac{z \sinh sz}{s} p \\ \frac{a^2}{2} \frac{z \sinh sz}{s} q \\ \frac{\sinh sz}{s} + \frac{a^2}{2} \left(\frac{\sinh sz}{s} - z \cosh sz \right) \end{pmatrix}$$

Similarly

$u_0' = 1$, $v_0' = 0$, $w_0 = 0$ gives

$$\begin{pmatrix} U^4 \\ \\ U^4 \\ \\ W^4 \end{pmatrix} = \begin{pmatrix} \frac{\sinh sz}{s} + \frac{a^2}{2(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)\frac{p^2}{s^2} \\ \\ \frac{a^2}{2(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)\frac{pq}{s^2} \\ \\ -\frac{a^2}{2(a^2+1)}\frac{z\sinh sz}{s}\,p \end{pmatrix}$$

$u_0' = 0$, $v_0' = 1$, $w_0 = 0$ gives

$$\begin{pmatrix} U^5 \\ \\ V^5 \\ \\ W^5 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)\frac{pq}{s^2} \\ \\ \frac{\sinh sz}{s} + \frac{a^2}{2(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)\frac{q^2}{s^2} \\ \\ -\frac{a^2}{a(a^2+1)}\frac{z\sinh sz}{s}\,q \end{pmatrix}$$

$u_0' = v_0' = 0$, $w_0 = 1$ gives

$$\begin{pmatrix} U^6 \\ \\ V^6 \\ \\ W^6 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{a(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)p \\ \\ \frac{a^2}{2(a^2+1)}\left(z\cosh sz - \frac{\sinh sz}{s}\right)q \\ \\ \cosh sz - \frac{a^2}{2(a^2+1)}\,sz\sinh sz \end{pmatrix}$$

It is evident from the B.C. at $z = 0$ that these six solutions are linearly independent and hence span the solution space of $L\,\overline{U} = (p^2 + q^2)\overline{U}$.

Solutions of $L\,\overline{U} - (p^2 + q^2)\overline{U} = \overline{F}(z)$.

The variation of constants formula will be used. For this purpose it is convenient to write the equation as a 1st order system. The equation has the form

$$A\,\overline{U}'' + B\,\overline{U}' - C_s\overline{U} = \overline{F}$$

where

$$C_s = s^2 1 - C = \begin{pmatrix} s^2 + a^2p^2 & a^2pq & 0 \\ a^2pq & s^2 + a^2q^2 & 0 \\ 0 & 0 & s^2 \end{pmatrix}$$

Now $A^T = A \geqslant 1$, whence

$$A = A^{1/2}\,A^{1/2}\ , \quad A^{1/2} = (A^{1/2})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{1+a^2} \end{pmatrix}$$

Thus

$$A^{1/2}\,\overline{U}'' + A^{-1/2}\,B\,\overline{U}' - A^{-1/2}\,C_s\,\overline{U} = A^{-1/2}\,\overline{F}$$

Put

$$\overline{V} = A^{1/2}\,\overline{U} = \begin{pmatrix} U_1 \\ U_2 \\ \sqrt{1+a^2}\;U_3 \end{pmatrix}$$

$$\overline{U} = A^{-1/2}\,\overline{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3/\sqrt{1+a^2} \end{pmatrix}$$

Then

$$\overline{V}'' + (A^{-1/2}\,B\,A^{-1/2})\overline{V}' - (A^{-1/2}\,C_s\,A^{-1/2})\overline{V} = A^{-1/2}\,\overline{F}$$

or

$$\overline{V}'' + B_A\,\overline{V}' - C_A\,\overline{V} = \overline{G}$$

where

$$B_A = A^{-1/2}\,B\,A^{-1/2} = -\,B_A^T$$

$$C_A = A^{-1/2}\,C_3\,A^{-1/2} = C_A^T$$

$$\overline{G} = A^{-1/2}\,\overline{F}$$

Explicitly,

$$B_A = \frac{a^2}{(1+a^2)^{1/2}} \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -q \\ p & q & 0 \end{pmatrix}$$

$$C_A = \begin{pmatrix} s^2 + a^2p^2 & a^2pq & 0 \\ a^2pq & s^2 + a^2q^2 & 0 \\ 0 & 0 & \dfrac{s^2}{1+a^2} \end{pmatrix}$$

A 1st order system equivalent to the above 2nd order system may be obtained by setting

$$Y = \overline{V} \ , \quad Z = \overline{V}'$$

$$X = \binom{Y}{Z} = \binom{\overline{V}}{\overline{V}'}$$

Then

$$Y' = \overline{V}' = Z \ , \quad Z' = \overline{V}'' = -B_A Z + C_A Y + \overline{G}$$

and

$$X' = \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix} X + \begin{pmatrix} 0 \\ \overline{G} \end{pmatrix}$$

or

$$X' = M X + H(z) \ , \quad H = \binom{0}{\overline{G}}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix}$$

A fundamental matrix for $X' = M X$ is a 6 x 6 matrix solution $\Phi(z)$ of

$$\begin{cases} \Phi'(z) = M \Phi(z) \\ \Phi(0) = 1 \end{cases}$$

An explicit representation of $\Phi(z)$ can be derived from the solution basis for $L \overline{U} = s^2 \overline{U}$. Indeed, each solution $\overline{U}^j = (U^j \ V^j \ W^j)^T$ of

$L\,\bar{U}^j = s^2\bar{U}^j$ gives a solution X^j of $X^{j\prime} = M\,X^j$, namely

$$X^j = \begin{pmatrix} \bar{V}^j \\ \bar{V}^{j\prime} \end{pmatrix} = \begin{pmatrix} A^{1/2}\,\bar{U}^j \\ A^{1/2}\,\bar{U}^{j\prime} \end{pmatrix}$$

Thus, in view of the B.C.'s at $z = 0$,

$$\Phi(z) = (X^1\; X^2\; (1+a^2)^{-1/2}X^6\; X^4\; X^5\; (1+a^2)^{-1/2}X^3)$$

$$= \begin{pmatrix} & M_2 & & & M_1 & \\ U^1 & U^2 & (1+a^2)^{-1/2}U^6 & U^4 & U^5 & (1+a^2)^{-1/2}U^3 \\ V^1 & V^2 & (\;)^{-1/2}V^6 & V^4 & V^5 & (\;)^{-1/2}V^3 \\ (1+a^2)^{1/2}W^1 & (\;)^{1/2}W^2 & W^6 & (1+a^2)^{1/2}W^4 & (1+a^2)^{1/2}W^5 & W^3 \\ U^1{}' & U^2{}' & (\;)^{-1/2}U^6{}' & U^4{}' & U^5{}' & (\;)^{-1/2}U^3{}' \\ V^1{}' & V^2{}' & (\;)^{-1/2}V^6{}' & V^4{}' & V^5{}' & (\;)^{-1/2}V^3{}' \\ (1+a^2)^{1/2}W^1{}' & (\;)^{1/2}W^2{}' & W^6{}' & (\;)^{1/2}W^4{}' & (\;)^{1/2}W^5{}' & W^3{}' \end{pmatrix}$$

where U^j, V^j, W^j are defined on pp. 16-17.

The fundamental matrix makes it possible to calculate a solution of $X' = M\,X + H$, namely

$$X(z) = \Phi(z)\,'\int_0^z \Phi^{-1}(\zeta)\;H(\zeta)d\zeta$$

Indeed,

$$X' = \Phi(z)\,'\int_0^z \Phi^{-1}(\zeta)\;H(\zeta)d\zeta + \Phi(z)\;\Phi^{-1}(z)\;H(z)$$

$$= M\,X + H$$

Moreover,

$$X(0) = 0$$

Thus the general solution of $X' = M X + H$ is given by

$$X(z) = \Phi(z) X_0 + \Phi(z) \int_0^z \Phi(\zeta)^{-1} H(\zeta) d\zeta$$

The direct calculation of $\Phi(\zeta)^{-1}$ is difficult, but note that if

$$\underline{P} = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} = \underline{P}^T$$

and

$$E(z) = \frac{1}{2} X(z)^T P X(z)$$

then $X' = M X \Rightarrow$

$$E'(z) = X(z)^T P X'(z) = X(z)^T PM X(z) \equiv 0$$

because

$$PM = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix} = \begin{pmatrix} 0 & C_A \\ -C_A & B_A \end{pmatrix}$$

and hence

$$(PM)^T = - PM$$

Thus $E(z) = \text{const.}$ $\forall$ solutions of $X' = M X$. Take $X(z) = \Phi(z) X_0$ ($X_0 \in \mathbb{R}^6$ arbitrary). Then

$$2E(z) = (\Phi(z) X_0)^T P \Phi(z) X_0 = X_0^T \Phi(z)^T P \Phi(z) X_0$$

$$= X_0^T P X_0 = 2E(0) \qquad \forall X_0 \in \mathbb{R}^6$$

It follows that

$$\Phi(z)^T P \Phi(z) = P \qquad \forall z \in \mathbb{R} .$$

Since P is non-singular

$$(P^{-1} \Phi(z)^T P) \Phi(z) = 1$$

whence

$$\Phi(z)^{-1} = \underline{P}^{-1} \Phi(z)^T \underline{P}$$

Thus

$$(*) \qquad X(z) = \Phi(z) X_0 + \Phi(z)\underline{P}^{-1} \int_0^z \Phi(\zeta)^T \underline{P} H(\zeta)d\zeta$$

<u>Solution of the B.V. Problem of p. 13.</u>

Recall that

$$X = \binom{\overline{V}}{\overline{V}'} = (U_1 \; U_2 \; (1+a^2)^{1/2} U_3 \; U_1' \; U_2' \; (1+a^2)^{1/2} U_3')^T$$

$$X_0 = (u_0 \; v_0 \; (1+a^2)^{1/2} w_0 \; u_1' \; v_0' \; (1+a^2)^{1/2} w_0')$$

Thus the solution (*) satisfies B.C.1 (at $z = 0$) $\Longleftrightarrow u_0' = v_0' = w_0 = 0$.

Thus if we write

$$X^{\pi}(z) = \Phi(z) \underline{P}^{-1} \int_0^z \Phi(\zeta)^T \underline{P} H(\zeta)d\zeta$$

then substituting in (*) gives

$$U_1(z) = \Phi_{11}(z)\, u_0 + \Phi_{12}(z)\, v_0 + (1+a^2)^{1/2}\, \Phi_{16}(z) w_0' + X_1^{\pi}(z)$$

$$U_2(z) = \Phi_{21}(z)\, u_0 + \Phi_{22}(z)\, v_0 + (1+a^2)^{1/2}\, \Phi_{26}(z)\, w_0' + X_2^{\pi}(z)$$

$$(1+a^2)^{1/2} U_3(z) = \Phi_{31}(z)\, u_0 + \Phi_{32}(z)\, v_0 + (1+a^2)^{1/2}\, \Phi_{36}(z)\, w_0' + X_3^{\pi}(z)$$

or (see p. 21) if $U^{\pi}(z) = X_1^{\pi}(z)$, $V^{\pi}(z) = X_2^{\pi}(z)$, $W^{\pi}(z) = (1+a^2)^{-1/2} X_3^{\pi}(z)$,

$$U(z) = U^1(z)\, u_0 + U^2(z)\, v_0 + U^3(z)\, w_0' + U^{\pi}(z)$$

$$V(z) = V^1(z)\, u_0 + V^2(z)\, v_0 + V^3(z)\, w_0' + V^{\pi}(z)$$

$$W(z) = W^1(z)\, u_0 + W^2(z)\, v_0 + W^3(z)\, w_0' + W^{\pi}(z)$$

Thus (p. 15)

$$U(z) = (\cosh sz)\, u_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (p^2 u_0 + pqv_0 + pw_0') + U^{\pi}(z)$$

$$V(z) = (\cosh sz)\, v_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (pqu_0 + q^2 v_0 + qw_0') + V^{\pi}(z)$$

$$W(z) = \frac{\sinh sz}{s} w_0' + \frac{a^2}{2} (\frac{\sinh sz}{s} - z \cosh sz)(pu_0 + qv_0 + w_0') + W^{\pi}(z)$$

To complete the solution of the B.V. problem of p. 13 the initial values u_0, v_0, w_0' must be chosen so that the B.C.2 at $z = h$ is satisfied. The derivatives U', V', W' are needed. They are given by

$$U'(z) = s(\sinh sz) u_0 + \frac{a^2}{2} (\frac{\sinh sz + sz \cosh sz}{s})(p^2 u_0 + pqv_0 + pw_0') + U^{\pi}{}'(z)$$

$$V'(z) = s(\sinh sz) v_0 + \frac{a^2}{2} (\frac{\sinh sz + sz \cosh sz}{s})(pqu_0 + q^2 v_0 + qw_0') + V^{\pi}{}'(z)$$

$$W'(z) = (\cosh sz) w_0' + \frac{a^2}{2} (-sz \sinh sz)(pu_0 + qv_0 + w_0') + W^{\pi}{}'(z)$$

Thus the B.C.2 (p. 13) gives

$U'(h) - p\,W(h)$

$$= s(\sinh sh)u_0 + \frac{a^2}{2}\left(\frac{\sinh sh + sh\cosh sh}{s}\right)(p^2u_0 + pqv_0 + pw_0')$$

$$- p\,\frac{\sinh sh}{s}\,w_0' - \frac{pa^2}{2}\left(\frac{\sinh sh}{s} - h\cosh sh\right)(pu_0 + qv_0 + w_0')$$

$$+ U^{\pi\prime}(h) - p\,W^{\pi}(h) = 0$$

$V'(h) - q\,W(h)$

$$= s(\sinh sh)v_0 + \frac{a^2}{2}\left(\frac{\sinh sh + sh\cosh sh}{s}\right)(pqu_0 + q^2v_0 + qw_0')$$

$$- q\,\frac{\sinh sh}{s}\,w_0' - \frac{qa^2}{2}\left(\frac{\sinh sh}{s} - h\cosh sh\right)(pu_0 + qv_0 + w_0')$$

$$+ V^{\pi\prime}(h) - q\,W^{\pi}(h) = 0$$

$p\,U(h) + q\,V(h) + \dfrac{a^2+1}{a^2-1}\,W'(h)$

$$= p(\cosh sh)u_0 + \frac{pa^2}{2}\,\frac{h\sinh sh}{s}\,(p^2u_0 + pqv_0 + pw_0')$$

$$+ q(\cosh sh)v_0 + \frac{qa^2}{2}\,\frac{h\sinh sh}{s}\,(pqu_0 + q^2v_0 + qw_0')$$

$$+ \left(\frac{a^2+1}{a^2-1}\right)(\cosh sh)w_0' + \left(\frac{a^2+1}{a^2-1}\right)\frac{a^2}{2}\,(-sh\sinh sh)(pu_0 + qv_0 + w_0')$$

$$+ p\,U^{\pi}(h) + q\,V^{\pi}(h) + \left(\frac{a^2+1}{a^2-1}\right)W^{\pi\prime}(h) = \left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0(h)$$

This is a system of linear equations for u_0, v_0, w_0' of the form

$$d_{11}u_0 + d_{12}v_0 + d_{13}w_0' = f_1(p,q)\,(= -U^{\pi\prime}(h) + p\,W^{\pi}(h))$$

$$d_{21}u_0 + d_{22}v_0 + d_{23}w_0' = f_2(p,q)\,(= -V^{\pi\prime}(h) + q\,W^{\pi}(h))$$

$$d_{31}u_0 + d_{32}v_0 + d_{33}w_0' = f_3(p,q)\,(= -p\,U^{\pi}(h) - q\,V^{\pi}(h) - \left(\frac{a^2+1}{a^2-1}\right)W^{\pi\prime}(h) + \left(\frac{2}{\pi}\right)^{1/2}\hat{v}_0(h))$$

where

$$d_{11} = s(\sinh sh) + \frac{a^2p^2}{2s}(\sinh sh + sh\cosh sh) - \frac{a^2p^2}{2s}(\sinh sh - sh\cosh sh)$$

$$= s(\sinh sh) + a^2p^2\, h\cosh sh$$

$$d_{12} = \frac{a^2}{2}\; pq\;(\frac{\sinh sh}{s} + h\cosh sh) - \frac{a^2}{2}\; pq\;(\frac{\sinh sh}{s} - h\cosh sh)$$

$$= a^2pq\; h\cosh sh$$

$$d_{13} = \frac{a^2}{2}\; p\;(\frac{\sinh sh}{s} + h\cosh sh) - \frac{p}{s}\sinh sh - \frac{a^2}{2}\, p\;(\frac{\sinh sh}{s} - h\cosh sh)$$

$$= a^2ph\cosh sh - p\,\frac{\sinh sh}{s}$$

$$d_{21} = \frac{a^2}{2}\; pq\;(\frac{\sinh sh}{s} + h\cosh sh) - \frac{a^2}{2}\; pq\;(\frac{\sinh sh}{s} - h\cosh sh)$$

$$= a^2pqh\cosh sh = d_{12}$$

$$d_{22} = s(\sinh sh) + \frac{a^2}{2}\, q^2\;(\frac{\sinh sh}{s} + h\cosh sh) - \frac{a^2}{2}\; q^2\;(\frac{\sinh sh}{s} - h\cosh sh)$$

$$= s(\sinh sh) + a^2q^2\; h\cosh sh$$

$$d_{23} = \frac{a^2}{2}\; q\;(\frac{\sinh sh}{s} + h\cosh sh) - q\,\frac{\sinh sh}{s} - \frac{a^2}{2}\; q\;(\frac{\sinh sh}{s} - h\cosh sh)$$

$$= -\frac{q}{s}\sinh sh + a^2qh\cosh sh$$

$$d_{31} = p\cosh sh + \frac{a^2}{2}\; p^3\,\frac{h}{s}\sinh sh + \frac{a^2}{2}\; pq^2\,\frac{h}{s}\sinh sh - (\frac{a^2+1}{a^2-1})\,\frac{a^2}{2}\; sh\; p\sinh sh$$

$$= p\cosh sh + (\frac{a^2}{2}\; psh - (\frac{a^2+1}{a^2-1})\,\frac{a^2}{2}\; psh)\sinh sh$$

$$= p\cosh sh + \frac{a^2}{2}\; psh\;(-\frac{2}{a^2-1})\sinh sh = p\cosh sh - \frac{a^2}{a^2-1}\; p\; sh\sinh sh$$

$$d_{32} = \frac{a^2}{2} p^2 q \frac{h \sinh sh}{s} + q \cosh sh + \frac{a^2}{2} q^3 \frac{h \sinh sh}{s} - \left(\frac{a^2+1}{a^2-1}\right) \frac{a^2}{2} sh\, q \sinh sh$$

$$= q \cosh sh + \frac{a^2}{2} qsh \sinh sh - \frac{a^2}{2} \left(\frac{a^2+1}{a^2-1}\right) qsh \sinh sh$$

$$= q \cosh sh + \frac{a^2}{2} qsh \sinh sh \left(\frac{-2}{a^2-1}\right)$$

$$= q \cosh sh - \frac{a^2}{a^2-1} qsh \sinh sh$$

$$d_{33} = \frac{a^2}{2} p^2 \frac{h \sinh sh}{s} + \frac{a^2}{2} q^2 \frac{h \sinh sh}{s} + \frac{a^2+1}{a^2-1} \cosh sh - \left(\frac{a^2+1}{a^2-1}\right) \frac{a^2}{2} hs \sinh sh$$

$$= \frac{a^2}{2} sh \sinh sh - \frac{a^2}{2} \left(\frac{a^2+1}{a^2-1}\right) sh \sinh sh + \frac{a^2+1}{a^2-1} \cosh sh$$

$$= \left(\frac{-2}{a^2-1}\right) \frac{a^2}{2} sh \sinh sh + \frac{a^2+1}{a^2-1} \cosh sh$$

$$= \frac{a^2+1}{a^2-1} \cosh sh - \frac{a^2}{a^2-1} sh \sinh sh$$

<u>The Cofactors of $Q = (d_{jk})$.</u>

Let $Q_{jk} = (\mathrm{cof} Q)_{jk}$

$$Q_{11} = d_{22} d_{33} - d_{32} d23$$

$$= (s \sinh sh + a^2 q^2 h \cosh sh)\left(\frac{a^2+1}{a^2-1} \cosh sh - \frac{a^2}{a^2-1} sh \sinh s\right)$$

$$+ \left(q \cosh sh - \frac{a^2}{a^2-1} q\, sh \sinh s\right)\left(+ \frac{q}{s} \sinh sh - a^2 qh \cosh sh\right)$$

$$= \frac{1}{a^2-1} [(a^2+1)s \sinh \cosh - a^2 s^2 h \sinh^2 + (a^2+1)a^2 q^2 h \cosh^2 - a^4 q^2 h^2 s \sinh \cos$$

$$+ q^2 [\frac{1}{s} \sinh \cosh - a^2 h \cosh^2 - \frac{a^2}{a^2-1} h \sinh^2 + \frac{a^4}{a^2-1} h^2 s \sinh \cosh]$$

$$= \left(\frac{(a^2+1)a^2 q^2 h}{a^2-\lambda} - a^2 q^2 h\right) \cosh^2 + \left(\frac{-a^2 s^2 h}{a^2-1} - \frac{a^2 q^2 h}{a^2-1}\right) \sinh^2$$

$$+ \left(\frac{a^2+1}{a^2-1} s - \frac{a^4 q^2 h^2 s}{a^2-1} + \frac{q^2}{s} + \frac{a^4 q^2 h^2 s}{a^2-1}\right) \sinh \cosh$$

$$= a^2q^2h \quad \frac{2}{a^2-1} \cosh^2 - \frac{a^2(s^2+q^2)h}{a^2-1} \sinh^2 + \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh \cosh$$

$$= \frac{a^2(s^2+q^2)h}{a^2-1} + \frac{2a^2q^2h - a^2(s^2+q^2)h}{a^2-1} \cosh^2 + \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh \cosh$$

$$= \frac{a^2(s^2+q^2)h}{a^2-1} - \frac{a^2p^2h}{a^2-1} \cosh^2 sh + 2 \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh 2sh$$

$$Q_{12} = - (d_{21}d_{33} - d_{31}d_{23}) = d_{31}d_{23} - d_{21}d_{33}$$

$$= (p \cosh - \frac{a^2}{a^2-1} psh \sinh)(- \frac{q}{s} \sinh + a^2qh \cosh)$$

$$- (a^2pqh \cosh)(\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh)$$

$$= - \frac{pq}{s} \sinh \cosh + a^2pqh \cosh^2 + \frac{a^2qph}{(a^2-1)} \sinh^2 - \frac{a^4pqsh^2}{a^2-1} \sinh \cosh$$

$$- \frac{a^2(a^2+1)}{a^2-1} pqh \cosh^2 + \frac{a^4pqsh^2}{a^2-1} \sinh \cosh$$

$$= - \frac{a^2pqh}{a^2-1} + (a^2pqh + \frac{a^2pqh}{a^2-1} - \frac{(a^4+a^2)pqh}{a^2-1}) \cosh^2$$

$$+ (- \frac{pq}{s}) \sinh \cosh$$

$$= - \frac{a^2pqh}{a^2-1} + a^2pqh (1 + \frac{1}{a^2-1} - \frac{a^2+1}{a^2-1}) \cosh^2 - \frac{pq}{s} \sinh \cosh$$

$$= - \frac{a^2pqh}{a^2-1} - \frac{a^2pqh}{a^2-1} \cosh^2 sh - \frac{pq}{2s} \sinh 2sh$$

$$Q_{13} = d_{21}d_{32} - d_{31}d_{22}$$

$$= (a^2pqh \cosh)(q \cosh - \frac{a^2}{a^2-1} q\ sh \sinh)$$

$$- (p \cosh - \frac{a^2}{a^2-1} psh \sinh)(s \sinh + a^2q^2h \cosh)$$

$$= (\cosh - \frac{a^2}{a^2-1} sh \sinh)(-ps \sinh)$$

$$= \frac{a^2}{a^2-1} phs^2 \sinh^2 sh - ps \sinh sh \cosh sh$$

$$= -\frac{a^2}{a^2-1} phs^2 + \frac{a^2}{a^2-1} phs^2 \cosh^2 sh - \frac{ps}{2} \sinh 2sh$$

$$Q_{21} = -(d_{12}d_{33} - d_{32}d_{13}) = d_{32}d_{13} - d_{12}d_{33}$$

$$= (q \cosh - \frac{a^2}{a^2-1} qsh \sinh)(a^2 ph \cosh - p \frac{\sinh}{s})$$

$$- (a^2 pqh \cosh)(\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh)$$

$$= a^2 pqh \cosh^2 - \frac{pq}{s} \sinh \cosh - \frac{a^4 pqsh^2}{a^2-1} \sinh \cosh + \frac{a^2 pqh}{a^2-1} \sinh^2$$

$$- \frac{a^2(a^2+1)pqh}{a^2-1} \cosh^2 + \frac{a^4 pqh^2 s}{a^2-1} \sinh \cosh$$

$$= -\frac{a^2 pqh}{a^2-1} + (a^2 pqh + \frac{a^2 pqh}{a^2-1} - \frac{a^2(a^2+1)pqh}{a^2-1}) \cosh^2$$

$$+ (-\frac{pq}{s}) \sinh \cosh$$

$$= -\frac{a^2 pqh}{a^2-1} - \frac{a^2 pqh}{a^2-1} \cosh^2 sh - \frac{pq}{2s} \sinh 2sh = Q_{12}$$

$$Q_{22} = d_{11}d_{33} - d_{31}d_{13}$$

$$= (s \sinh + a^2 p^2 h \cosh)(\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh)$$

$$- (p \cosh - \frac{a^2}{a^2-1} psh \sinh)(a^2 ph \cosh - \frac{p}{s} \sinh)$$

$$= \frac{s(a^2+1)}{a^2-1} \sinh \cosh - \frac{a^2 s^2 h}{a^2-1} \sinh^2 + \frac{a^2(a^2+1)p^2 h}{a^2-1} \cosh^2 - \frac{a^4 p^2 sh^2}{a^2-1} \sinh \cosh$$

$$- a^2 p^2 h \cosh^2 + \frac{p^2}{s} \sinh \cosh + \frac{a^4 p^2 sh^2}{a^2-1} \sinh \cosh - \frac{a^2 p^2 h}{a^2-1} \sinh^2$$

$$= -\frac{a^2(s^2+p^2)h}{a^2-1}(\cosh^2 - 1) + (\frac{a^2(a^2+1)p^2 h}{a^2-1} - a^2 p^2 h) \cosh^2$$

$$+ (\frac{s(a^2+1)}{a^2-1} - \frac{p^2}{s}) \sinh \cosh$$

$$= \frac{a^2(p^2+s^2)h}{a^2-1} + a^2 h (\frac{(a^2+1)p2}{a^2-1} - p^2 - \frac{s^2+p^2}{a^2-1}) \cosh^2$$

$$+ \frac{s^2(a^2+1) + p^2(a^2-1)}{s(a^2-1)} \sinh \cosh$$

$$= \frac{a^2(p^2+s^2)h}{a^2-1} - \frac{a^2q^2h}{a^2-1} \cosh^2 sh + \frac{a^2(s^2+p^2)+q^2}{2s(a^2-1)} \sinh 2sh$$

$$Q_{23} = -(d_{11}d_{32} - d_{31}d_{12}) = d_{31}d_{12} - d_{11}d_{32}$$

$$= (\cosh - \frac{a^2}{a^2-1} sh \sinh)(a^2p^2qh \cosh)$$

$$- q\ (s \sinh + a^2p^2h \cosh)(\cosh - \frac{a^2}{a^2-1} sh \sinh)$$

$$= (\cosh - \frac{a^2}{a^2-1} sh \sinh)(-\ qs \sinh)$$

$$= -\ qs \sinh \cosh + \frac{a^2s^2qh}{a^2-1} \sinh^2$$

$$= -\ \frac{a^2s^2qh}{a^2-1} + \frac{a^2s^2qh}{a^2-1} \cosh^2 sh - \frac{qs}{2} \sinh 2sh$$

$$Q_{31} = d_{12}d_{23} - d_{22}d_{13}$$

$$= (a^2pqh \cosh)(-\ \frac{q}{s} \sinh + a^2qh \cosh)$$

$$-\ (s \sinh + a^2q^2h \cosh)(a^2ph \cosh - \frac{p}{s} \sinh)$$

$$-\ a^2phs \sinh \cosh + p \sinh^2$$

$$= -p + p \cosh^2 sh - \frac{a^2phs}{2} \sinh 2sh$$

$$Q_{32} = -\ (d_{12}d_{23} - d_{21}d_{13}) = d_{21}d_{13} - d_{12}d_{23}$$

$$= (a^2pqh \cosh)(a^2ph \cosh - \frac{p}{s} \sinh)$$

$$-\ (a^2pqh \cosh)(\ \frac{q}{s} \sinh + a^2qh \cosh)$$

$$= (a^2pqh \cosh)(a^2h(p-q) \cosh + \frac{q-p}{s} \sinh)$$

$$= (a^2pqh \cosh sh)(p-q)(a^2h \cosh - \frac{1}{s} \sinh)$$

$$Q_{33} = d_{11}d_{22} - d_{21}d_{12}$$

$$= (s \sinh + a^2p^2h \cosh)(s \sinh + a^2q^2h \cosh) - a^4p^2q^2h^2 \cosh^2$$

$$= s^2 \sinh^2 + (a^2p^2sh + a^2q^2sh) \sinh \cosh$$

$$= s^2 \sinh^2 sh + a^2s^3h \sinh sh \cosh sh$$

$|Q| = \det(d_{jk})$ can be calculated from Luré, p. 153 and the correspondence (see p. 14)

$$ip \Leftrightarrow \alpha_1 \, , \quad iq \Leftrightarrow \alpha_2 \, , \quad is \Leftrightarrow D$$

This gives

$$|Q| = 2a^2h(is)^3 \sin(ish)(1 + \frac{\sin 2ish}{2ish})$$

$$= 2a^2h\, s^3 \sinh sh \, (1 + \frac{\sinh 2sh}{2sh})$$

Solution of the System on p. 25. We can solve by means of the relations.

$$\sum_{k=1}^{3} Q_{kj}d_{k\ell} = |Q|\delta_{j\ell}$$

Thus

$$(*)\quad \begin{cases} |Q|\; u_0 = Q_{11}f_1 + Q_{21}f_2 + Q_{31}f_3 \\ |Q|\; v_0 = Q_{12}f_1 + Q_{22}f_2 + Q_{32}f_3 \\ |Q|\; w_0' = Q_{13}f_1 + Q_{23}f_2 + Q_{33}f_3 \end{cases}$$

The only real zero of $|Q(s)|$ is at $s = \sqrt{p^2+q^2} = 0$. Thus $\forall$ real $(p,q) \neq (0,0)$

$$u_0(p,q) = \sum_{j=1}^{3} \frac{Q_{j1}(p,q)f_j(p,q)}{|Q(s)|}$$

$$v_0(p,q) = \sum_{j=1}^{3} \frac{Q_{j2}(p,q)f_j(p,q)}{|Q(s)|}$$

$$w_0'(p,q) = \sum_{j=1}^{3} \frac{Q_{j3}(p,q)f_j(p,q)}{|Q(s)|} .$$

Substituting for u_0, v_0, w_0' in the equations on p. 24 gives

$$U(p,q,z)\ ,\quad V(p,q,z)\ ,\quad W(p,q,z)$$

Residue Series Representation for $\hat{u}(p,y,z)$, $\hat{v}(p,y,z)$, $\hat{w}(p,y,z)$.

The equations on pp. 4-8 give

$$\hat{u}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, \tilde{u}(p,q,z)\,dq = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, U(p,q,z)\,dq$$

$$\hat{v}(p,y,z) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, V(p,q,z)\,dq$$

$$\hat{w}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, W(p,q,z)\,dq$$

In particular,

$$\hat{u}(p,q,0) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, u_0(p,q)\,dq$$

$$\hat{v}(p,y,0) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, v_0(p,q)\,dq$$

$$\frac{\partial \hat{w}(p,y,0)}{\partial z} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq}\, w_0'(p,q)\,dq$$

Now the equations on p. 32 and p. 24 give U,V,W,u_0,v_0,w_0' as meromorphic functions of q for each fixed p. Thus residue series for the above functions can be obtained by deforming the contour in the upper half of the q-plane for $y > 0$ (lower half for $y < 0$). The poles of the integrals $U,\ldots,w_0'$ are the zeros of $|Q(s)|$. The cofactors $Q_{jk}(p,q)$ are holomorphic in the q-plane. Examinations of the formulas for U^{π},V^{π},W^{π} and $f_j(p,q)$ shows that these functions are analytic everywhere except at $s = 0$, because $P^{-1} = O(s^{-2})$. Thus special care is necessary in calculating the residue at $s = 0$ $(q = \pm i|p|)$.

<u>Zeros of $|Q(s)|$</u>.

There are two families of zeros

1) $\sinh sh = -i \sin(ish) = 0 \Leftrightarrow ish = is_n h = n\pi$, $n = 0,1,2$,

Thus

$$s_n = \sqrt{p^2+q_n^2} = \frac{n\pi}{ih} = -i\,\frac{n\pi}{h}, \quad p^2 + q_n^2 = -\left(\frac{n\pi}{h}\right)^2$$

$$q_n^2 = -\left(p^2 + \left(\frac{n\pi}{h}\right)^2\right), \quad q_n = i\sqrt{p^2+(n\pi/h)^2} = i\sqrt{p^2+\alpha_n^2}$$

$$q_0 = i\,|p|$$

These are simple zeros of $|Q(s)|$ for $n \geqslant 1$. However

$$|Q(s)| = \mathcal{O}(s^4) , \quad s \to 0$$

$$(p^2 + q^2)^2 = (q - i|p|)^2(q + i|p|)^2 \sim (2i|p|)^2(q - i|p|)^2$$

Thus q_0 is, in general, a higher-order pole.

2) $1 + \frac{\sin 2sh}{2sh} = 1 - i \frac{\sin 2ish}{2sh} = -i \left(\frac{2ish + \sin 2ish}{2sh}\right) = 0$

$\Longleftrightarrow 2ish = 2is_\nu h = 2\beta_\nu h \Longleftrightarrow s_\nu = \sqrt{p^2+q_\nu^2} = -i\beta_\nu$

$p^2 + q_\nu^2 = - \beta_\nu^2 , \quad q_\nu^2 = - (p^2 + \beta_\nu^2) , \quad q_\nu = i \sqrt{p^2+\beta_\nu^2}$

Calculation of $\overline{F}(z)$.

$\overline{F}(z)$ is defined on p. 13. Now on $y = 0\pm$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\tau_{zy} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

Thus

$$\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\partial u(x,0+,z)}{\partial y} = - \left(\frac{\partial v}{\partial x}\right)_0$$

$$\mathcal{F}_s\left(\frac{\partial u}{\partial y}\right)_0 = - \mathcal{F}_s\left(\frac{\partial v}{\partial x}\right)_0 = + p\, \mathcal{F}_c\, v_0$$

$$\mathcal{F}_c\left(\frac{\partial w}{\partial y}\right)_0 = - \mathcal{F}_c\left(\frac{\partial v}{\partial z}\right)_0$$

Thus

$$(\tfrac{\pi}{2})^{1/2} F_1(z) = p\ (1-a^2)\ F_c v_0$$

$$(\tfrac{\pi}{2})^{1/]} F_2(z) = -q\ (1+a^2)\ F_c v_0$$

$$(\tfrac{\pi}{2})^{1/2} F_3(z) = -\ (1-a^2)\ F_c\ (\tfrac{\partial v}{\partial z})_0$$

<u>Calculation of $U^{\pi}, V^{\pi}, W^{\pi}, U^{\pi\prime}, V^{\pi\prime}, W^{\pi\prime}$.</u>

From pp. 18-19.

$$H(z) = \begin{pmatrix} 0 \\ \overline{G(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1/2}\overline{F}(z) \end{pmatrix}$$

$$PH(\zeta) = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \overline{G(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{G}(z) \end{pmatrix}$$

Write

$$\Phi(z) = \begin{pmatrix} \Phi^{11}(z) & \Phi^{12}(z) \\ \Phi^{21}(z) & \Phi^{22}(z) \end{pmatrix}$$

Then

$$\Phi^T(\zeta) = \begin{pmatrix} \Phi^{11T}(\zeta) & \Phi^{21T}(\zeta) \\ \Phi^{12T}(\zeta) & \Phi^{22T}(\zeta) \end{pmatrix}$$

Hence

$$\Phi^T(\zeta)\, PH(\zeta) = \begin{pmatrix} \Phi^{11T} & \Phi^{21T} \\ \Phi^{12T} & \Phi^{22T} \end{pmatrix} \begin{pmatrix} 0 \\ -\overline{G} \end{pmatrix} = - \begin{pmatrix} \Phi^{21T}\overline{G} \\ \Phi^{22T}\overline{G} \end{pmatrix},$$

$$\underline{P}^{-1}\, \Phi^T\, PH = - \begin{pmatrix} C_A^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Phi^{21T}\overline{G} \\ \Phi^{22T}\overline{G} \end{pmatrix} = - \begin{pmatrix} C_A^{-1}\Phi^{21T}\overline{G} \\ -\ \Phi^{22T}\overline{G} \end{pmatrix},$$

$$\Phi(z)\ P^{-1}\ \Phi^T(\zeta)\ P\ H(\zeta)$$

$$= \begin{pmatrix} \Phi^{11}(z) & \Phi^{12}(z) \\ \Phi^{21}(z) & \Phi^{22}(z) \end{pmatrix} \begin{pmatrix} -\ C_A^{-1}\Phi^{21T}(\zeta)\bar{G}(\zeta) \\ \Phi^{22T}(\zeta)\bar{G}(\zeta) \end{pmatrix}$$

$$= \begin{pmatrix} -\Phi^{11}(z)C_A^{-1}\Phi^{21T}(\zeta)\bar{G}(\zeta) + \Phi^{12}(z)\Phi^{22T}(\zeta)\bar{G}(\zeta) \\ -\Phi^{21}(z)C_A^{-1}\Phi^{21T}(\zeta)\bar{G}(\zeta) + \Phi^{22}(z)\Phi^{22T}(\zeta)\bar{G}(\zeta) \end{pmatrix}$$

Thus (pp. 23-24)

$$\bar{U}^{\pi}(z) = (U^{\pi}(z), V^{\pi}(z), W^{\pi}(z))^T = A^{-1/2}(X_1^{\pi}, X_2^{\pi}, X_3^{\pi})^T$$

$$= \int_0^z A^{-1/2}(\Phi^{12}(z)\Phi^{22T}(\zeta) - \Phi^{11}(z)C_A^{-1}\Phi^{21T}(\zeta))A^{-1/2}\bar{F}(\zeta)d\zeta$$

$$A^{-1/2}\Phi^{11}(z) = (\bar{U}^1\bar{U}^2\bar{U}^6)A^{-1/2}$$

$$A^{-1/2}\Phi^{12}(z) = (\bar{U}^4\bar{U}^5\bar{U}^3)A^{-1/2}$$

$$\Phi^{21T}(\zeta) = (A^{1/2}(\bar{U}^{1\prime}\bar{U}^{2\prime}\bar{U}^{6\prime})A^{-1/2})^T = A^{-1/2}(\bar{U}^{1\prime}\bar{U}^{2\prime}\bar{U}^{6\prime})^T A^{1/2}$$

$$\Phi^{22T}(\zeta) = (A^{1/2}(\bar{U}^{4\prime}\bar{U}^{5\prime}\bar{U}^{3\prime})A^{-1/2})^T = A^{-1/2}(\bar{U}^{4\prime}\bar{U}^{5\prime}\bar{U}^{3\prime})^T A^{1/2}$$

$$A^{-1/2}\Phi^{12}(z)\Phi^{22T}(\zeta)A^{-1/2} = (\bar{U}^4(z)\bar{U}^5(z)\bar{U}^3(z))A^{-1}(\bar{U}^{4\prime}(\zeta)\bar{U}^{5\prime}(\zeta)\bar{U}^{3\prime}(\zeta))^T$$

$$A^{-1/2}C_A^{-1}A^{-1/2} = \frac{1}{s^4(1+a^2)} \begin{pmatrix} s^2+a^2q^2 & -a^2pq & 0 \\ -a^2pq & s^2+a^2p^2 & 0 \\ 0 & 0 & s^2(1+a^2) \end{pmatrix} = C_s^{-1}$$

$$A^{-1/2}\Phi^{11}(z)C_A^{-1}\Phi^{21T}(\zeta)A^{-1/2} = (\bar{U}^1(z)\bar{U}^2(z)\bar{U}^6(z))C_s^{-1}(\bar{U}^{1\prime}(\zeta)\bar{U}^{2\prime}(\zeta)\bar{U}^{6\prime}(\zeta))^T$$

Put

$$M_1(z) = (\overline{U}^4(z)\overline{U}^5(z)\overline{U}^3(z)) \;, \quad M_2(z) = (\overline{U}^1(z)\overline{U}^2(z)\overline{U}^6(z))$$

Then

$$\overline{U}^{\pi}(z) = \int_0^z \{M_1(z)A^{-1}M_1'(\zeta)^T - M_2(z)C_s^{-1}M_2'(\zeta)^T\}\overline{F}(\zeta)d\zeta$$

$$\overline{U}^{\pi}{}'(z) = \{M_1(z)A^{-1}M_1'(z)^T - M_2(z)C_s^{-1}M_2'(z)^T\}\ \overline{F}(z)$$

$$+ \int_0^z \{M_1'(z)A^{-1}M_1'(\zeta)^T - M_2'(z)C_s^{-1}M_2'(\zeta)^T\}\ \overline{F}(\zeta)d\zeta$$

Similarly,

$$\overline{U}^{\pi}{}'(z) = A^{-1/2}(X_4^{\pi}X_5^{\pi}X_6^{\pi})^T$$

$$= \int_0^z A^{-1/2}(\Phi^{22}(z)\Phi^{22T}(\zeta) - \Phi^{21}(z)C_A^{-1}\Phi^{21T}(\zeta))A^{-1/2}\ \overline{F}(\zeta)d\zeta$$

$$A^{-1/2}\Phi^{22}(z)\Phi^{22T}(\zeta)A^{-1/2} = (\overline{U}^{4}{}'(z)\overline{U}^{5}{}'(z)\overline{U}^{3}{}'(z))A^{-1}(\overline{U}^{4}{}'(\zeta)\overline{U}^{5}{}'(\zeta)\overline{U}^{3}{}'(\zeta))^T$$

$$A^{-1/2}\Phi^{21}(\zeta)C_A^{-1}\Phi^{21T}(\zeta)A^{-1/2} = (\overline{U}^{1}{}'(z)\overline{U}^{2}{}'(\zeta)\overline{U}^{6}{}'(z))A^{-1/2}C_A^{-1}A^{-1/2}(\overline{U}^{1}{}'(\zeta)\overline{U}^{2}{}'(\zeta)\overline{U}^{6}{}'(\zeta))^T$$

$$\overline{U}^{\pi}{}'(z) = \int_0^z \{M_1'(z)A^{-1}M_1'^T(\zeta) - M_2'(z)C_s^{-1}M_2'^T(\zeta)\}\ F(\zeta)d\zeta$$

$$\overline{U}^{\pi}{}''(z) = \{M_1'(z)A^{-1}M_1'(z)^T - M_2'(z)C_s^{-1}M_2'(z)^T\}\ \overline{F}(z)$$

$$+ \int_0^z \{M_1''(z)A^{-1}M_1'^T(\zeta) - M_2''(z)C_s^{-1}M_2'(\zeta)^T\}\ \overline{F}(\zeta)d\zeta$$

$$A\,\overline{U}^{\pi}{}''(z) + B\,\overline{U}^{\pi}{}'(z) - C_s\overline{U}^{\pi}(z) = A\{M_1'(z)A^{-1}M_1'(z)^T - M_2'(z)C_s^{-1}M_2'(z)^T\}\ \overline{F}(z)$$

$$= 1$$

An alternative derivation is as follows.

Try $$\overline{U}^{\pi}(z) = M_1(z)\overline{C}_2(z) + M_2(z)\overline{C}_1(z)$$

$$\overline{U}^{\pi\prime}(z) = M_1'(z)\overline{C}_2(z) + M_2'(z)\overline{C}_1(z)$$

$$+ M_1(z)\overline{C}_2'(z) + M_2(z)\overline{C}_1'(z) \leftarrow \text{set} = 0$$

$$\overline{U}^{\pi\prime\prime}(z) = M_1''(z)\overline{C}_2(z) + M_2''(z)\overline{C}_1(z)$$

$$+ M_1'(z)\overline{C}_2'(z) + M_2'(z)\overline{C}_1'(z)$$

$$A\,\overline{U}^{\pi\prime\prime} + B\,\overline{U}^{\pi\prime} - C_s\,\overline{U}^{\pi} = A\,M_1'(z)\overline{C}_2'(z) + A\,M_2'(z)\overline{C}_1'(z) = \overline{F}(z)$$

Thus

$$M_1(z)\overline{C}_2'(z) + M_2(z)\overline{C}_1'(z) = \overline{0}$$

$$M_1'(z)\overline{C}_2'(z) + M_2'(z)\overline{C}_1'(z) = A^{-1}\,\overline{F}(z)$$

or

$$\begin{pmatrix} U^1 & U^2 & U^6 & U^4 & U^5 & U^3 \\ \overline{U}^{1\prime} & \overline{U}^{2\prime} & \overline{U}^{6\prime} & \overline{U}^{4\prime} & \overline{U}^{5\prime} & \overline{U}^{3\prime} \end{pmatrix} \begin{pmatrix} \overline{C}_1' \\ \overline{C}_2' \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}\overline{F} \end{pmatrix}$$

or

$$\Phi^0(z) \begin{pmatrix} \overline{C}_1' \\ \overline{C}_2' \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}\overline{F} \end{pmatrix}$$

Note that

$$\Phi(z) = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \Phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} .$$

Thus (p. 22)

$$P = \begin{pmatrix} A^{-1/2}C_sA^{-1/2} & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \Phi^0(z)^T \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} A^{-1/2}C_sA^{-1/2} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \Phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \Phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \Phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix}$$

and

$$\begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} P \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} = \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

$$\Phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \Phi^0(z) = \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

or

$$\Phi^0(z)^{-1} = \begin{pmatrix} C_s^{-1} & 0 \\ 0 & -A^{-1} \end{pmatrix} \Phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

Applying this to the system on p. 38 gives

$$\Phi^0(z) \begin{pmatrix} \overline{C}_1' \\ \overline{C}_2' \end{pmatrix} = \begin{pmatrix} M_2(z) & M_1(z) \\ M_2'(z) & M_1'(z) \end{pmatrix} \begin{pmatrix} \overline{C}_1' \\ \overline{C}_2' \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}\overline{F} \end{pmatrix}$$

$$\begin{pmatrix} \overline{C}_1' \\ \overline{C}_2' \end{pmatrix} = \begin{pmatrix} C_s^{-1} & 0 \\ 0 & -A^{-1} \end{pmatrix} \begin{pmatrix} M_2^T(z) & M_2'^T(z) \\ M_1^T(z) & M_1'^T(z) \end{pmatrix} \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} 0 \\ A^{-1}\overline{F} \end{pmatrix}$$

$$\begin{pmatrix} \overline{C}_1'(z) \\ \\ \overline{C}_2'(z) \end{pmatrix} = \begin{pmatrix} C_s^{-1} & 0 \\ \\ 0 & -A^{-1} \end{pmatrix} \begin{pmatrix} -\ M_2'^{T}(z)\ \overline{F}(z) \\ \\ -\ M_1'^{T}(z)\ \overline{F}(z) \end{pmatrix}$$

$$= \begin{pmatrix} -\ C_s^{-1} M_2'^{T}(z)\ \overline{F}(z) \\ \\ A^{-1} M_1'^{T}(z)\ \overline{F}(z) \end{pmatrix}$$

$$\overline{U}^{\pi}(z) = \int_0^z \{M_1(z)A^{-1}M_1'^{T}(\zeta) - M_2(z)C_s^{-1}M_2'^{T}(\zeta)\}\ \overline{F}(\zeta)d\zeta$$

$$\Phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \Phi^0(z)$$

$$= \begin{pmatrix} M_2^T(z) & M_2'^{T}(z) \\ \\ M_1^T(z) & M_1'^{T}(z) \end{pmatrix} \begin{pmatrix} C_s & 0 \\ \\ 0 & -A \end{pmatrix} \begin{pmatrix} M_2(z) & M_1(z) \\ \\ M_2'(z) & M_1'(z) \end{pmatrix} = \begin{pmatrix} C_s & 0 \\ \\ 0 & -A \end{pmatrix}$$

$$= \begin{pmatrix} M_2^T & M_2'^{T} \\ \\ M_1^T & M_1'^{T} \end{pmatrix} \begin{pmatrix} C_sM_2 & C_sM_1 \\ \\ -AM_2' & -AM_1' \end{pmatrix}$$

$$= \begin{pmatrix} M_2^TC_sM_2 - M_2'^{T}AM_2' & M_2^TC_sM_1 - M_2'^{T}AM_1' \\ \\ M_1^TC_sM_2 - M_1'^{T}AM_2' & M_1^TC_sM_1 - M_1'^{T}AM_1' \end{pmatrix}$$

$$\equiv \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

Calculation of Coefficients in the Residue Series.

$$M_2(p,q,z) = \begin{pmatrix} U^1 & U^2 & U^6 \\ V^1 & V^2 & V^6 \\ W^1 & W^3 & W^6 \end{pmatrix}, \quad M_1(p,q,z) = \begin{pmatrix} U^4 & U^5 & U^3 \\ V^4 & V^5 & V^3 \\ W^4 & W^5 & W^3 \end{pmatrix}$$

The Poles $q_n = i\sqrt{p^2+\alpha_n^2}$, $s_n = -i\,\alpha_n$, $n = 1,2,3,\ldots (\alpha_n = \frac{n\pi}{h})$

$$\cosh s_n h = \cos i s_n h = \cosh \alpha_n h = \cos n\pi = (-1)^n$$

$$\sinh s_n h = -i \sin i s_n h = -i \sin n\pi = 0$$

$$M_2(p,q_n,h) = \begin{pmatrix} (-1)^n & 0 & \frac{a^2 hp}{2(a^2+1)}(-1)^n \\ 0 & (-1)^n & \frac{a^2 hq_n}{2(a^2+1)}(-1)^n \\ -\frac{a^2 h}{2}(-1)^n p & -\frac{a^2 h}{2}(-1)^n q_n & (-1)^n \end{pmatrix}$$

$$M_1(p,q_n,h) = \begin{pmatrix} \frac{a^2 hp^2}{2(a^2+1)(-\alpha_n^2)}(-1)^n & \frac{a^2 hpq_n(-1)^n}{2(a^2+1)(-\alpha_n^2)} & 0 \\ \frac{a^2 hpq_n}{2(a^2+1)(-\alpha_n^2)}(-1)^n & \frac{a^2 hq_n^2(-1)^n}{2(a^2+1)(-\alpha_n^2)} & 0 \\ 0 & 0 & -\frac{a^2 h}{2}(-1)^n \end{pmatrix}$$

$$M_2'(p,q_n,h) = \begin{pmatrix} \frac{a^2 hp^2}{2}(-1)^n & \frac{a^2 hpq_n}{2}(-1)^n & 0 \\ \frac{a^2 hpq_n}{2}(-1)^n & \frac{a^2 hp_n^2}{2}(-1)^n & 0 \\ 0 & 0 & -\frac{a^2(-\alpha_n^2)}{2(a^2+1)}(-1)^n \end{pmatrix}$$

$$M_1'(p,q_n,h) = \begin{pmatrix} (-1)^n & 0 & \frac{a^2}{2}hp(-1)^n \\ 0 & (-1)^n & \frac{a^2}{2}hq_n(-1)^n \\ -\frac{a^2}{2(a^2+1)}hp(-1)^n & -\frac{a^2}{2(a^2+1)}hq_n(-1)^n & (-1)^n \end{pmatrix}$$

For a simple pole at $q = q_0 = |p|$

$$\operatorname*{Res}_{q_0} \{e^{iyq} u_0(p,q)\} = \lim_{q\to q_0} \{(q-q_0)\, e^{iyq} u_0(p,q)\}$$

$$= e^{-y|p|} \lim_{q\to q_0} \{(q-q_0)\, u_0(p,q)\}$$

For a double pole

$$\operatorname*{Res}_{q_0} \{e^{iyq} u_0(p,q)\} = \lim_{q\to q_0} \frac{\partial}{\partial q} \{(q-q_0)^2 e^{iyq} u_0(p,q)\}$$

$$= \lim_{q\to q_0} [e^{iyq} \frac{\partial}{\partial q} \{(q-q_0)^2 u_0\} + iy\, e^{iyq} \{(q-q_0)^2 u_0\}]$$

$$= e^{iy|p|} \operatorname*{Res}_{q_0} u_0(p,q) + iy\, e^{-y|p|} a_{-2}(u_0)$$

For higher order poles, correspondingly higher order powers of y appear.

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