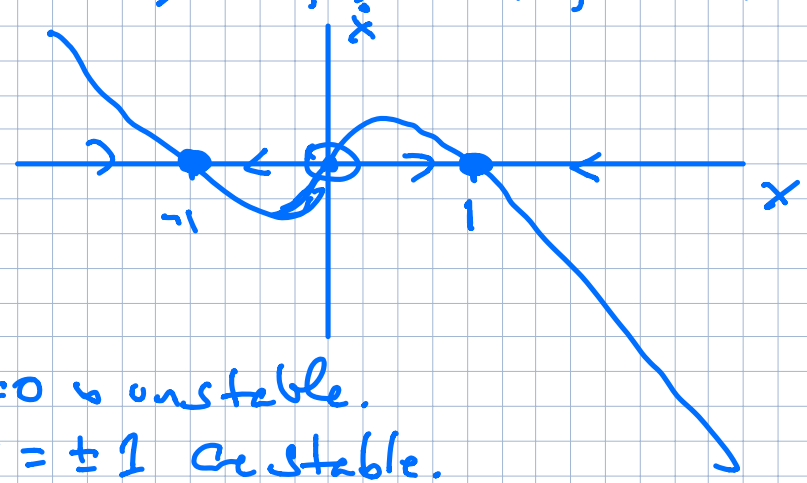
Quiz 1: For $\dot{x} = x - x^3$, sketch vector field, find fixed pts and determine their stability. Sketch sol'ns $x(t)$ for several initial conditions.

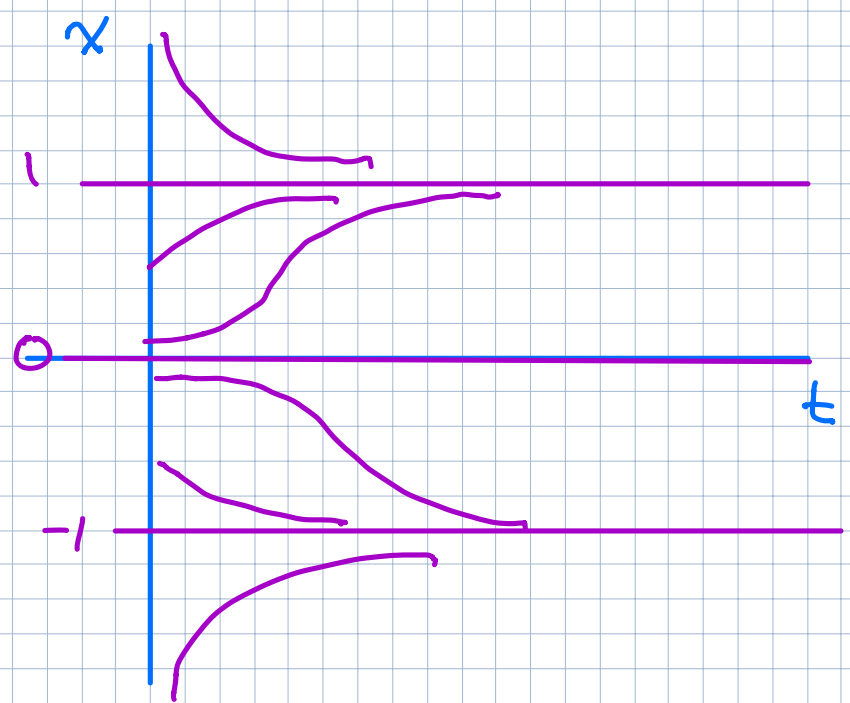
Sol'n: $\dot{x} = f(x) = x - x^3 = x(1-x^2) = x(1-x)(1+x)$.

FPs at $x=0$, $x=-1$, $x=1$



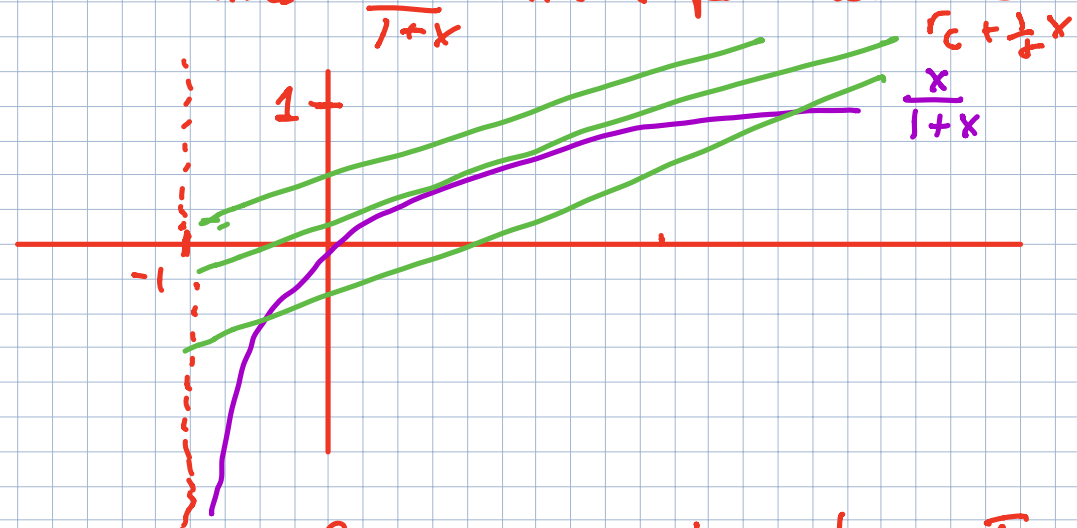
$x^* = 0$ is unstable.

$x^* = \pm 1$ are stable.



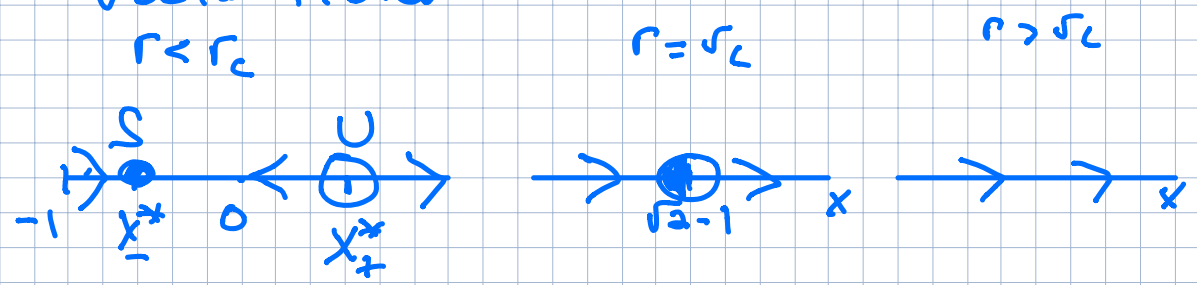
Quiz 2: For $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$ $(x > -1)$, sketch qualitatively different vector fields as r is varied. Show that a saddle-node bifurcation occurs when $r = r_c$ to be determined. Sketch bifurcation diagram.

Sol'n: $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$ has fixed points when $r + \frac{1}{2}x = \frac{x}{1+x}$. Note $r + \frac{1}{2}x$ has slope $\frac{1}{2}$ at $x=0$ while $\frac{x}{1+x}$ has slope 1 at $x=0$



For $r > r_c$, there are no intersections. For $r < r_c$ there are two intersections $x_-^* < 0 < x_+^*$. So a SN bifurcation occurs at $r = r_c$

Vector fields:



At $r = r_c$ and fixed point x, we have $f(x) = 0$ and $f'(x) = 0$.

(A) $$r + \frac{1}{2}x - \frac{x}{1+x} = 0$$

(B) $$\frac{1}{2} - \frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2} = \frac{1}{2} - \frac{1}{(1+x)^2} = 0$$

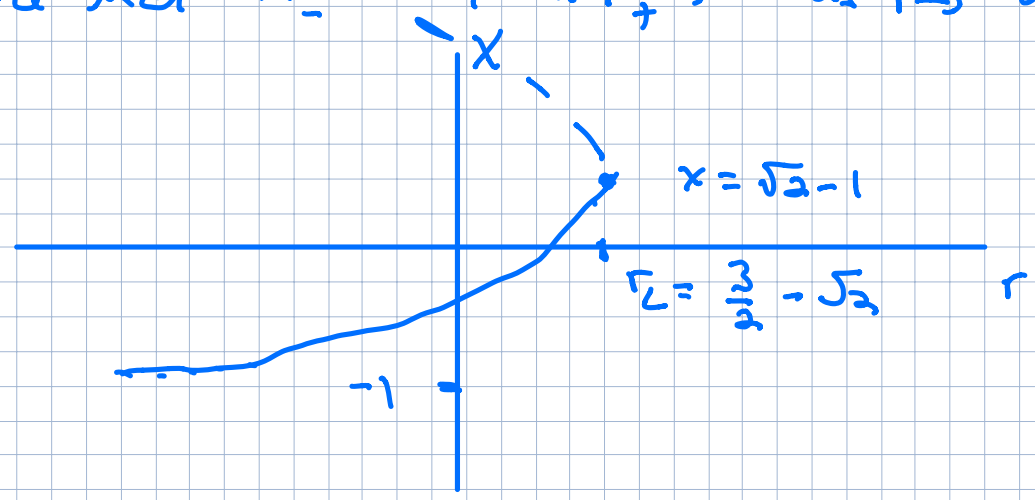
From (B) we see that we need $(1+x)^2 = 2$ so $x = \pm\sqrt{2} - 1$. Only $x = \sqrt{2} - 1$ is allowed. From (A), using $x = \sqrt{2} - 1$, we see

$$r = \frac{\sqrt{2}-1}{1+\sqrt{2}-1} - \frac{1}{2}(\sqrt{2}-1) \Rightarrow$$

$$r = r_c = \frac{\sqrt{2}-1}{\sqrt{2}} - \frac{\sqrt{2}-1}{2} = \frac{3}{2} - \sqrt{2}.$$

So saddle bifurcation occurs at $r = \frac{3}{2} - \sqrt{2} > 0$.

To help sketch the bifurcation diagram, note that $x_-^* \to -1$ and $x_+^* \to \infty$ as $r \to -\infty$.



Quiz 3: For $\dot{x} = rx - \frac{x}{1+x^2}$, sketch all qualitatively different vector fields, determine value(s) r_c at which bifurcations occur, classify the bifurcation type and sketch a bifurcation diagram.

Sol'n: $\dot{x} = x\left(r - \frac{1}{1+x^2}\right) = f(x)$

$f(x) = 0$ at $x=0$ so $x^* = 0$ is a fixed point $\forall r$.

There are additional fixed pts if $r - \frac{1}{1+x^2} = 0$. This happens for

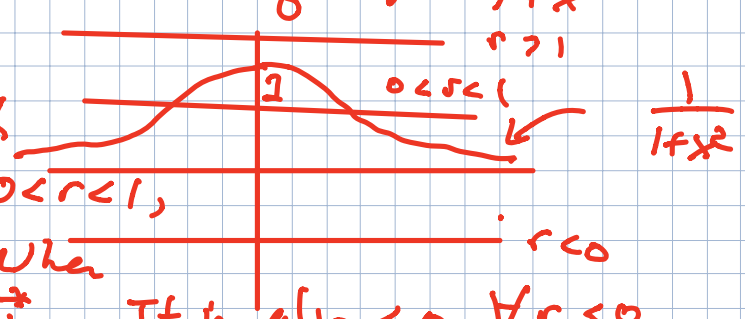
$x^*_\pm \equiv \pm\sqrt{\frac{1}{r} - 1}$ when this makes sense.

It makes sense if $0 < \frac{1}{r} - 1$. If $r \le 0$, this is not true. For $r > 0$, it is true if $0 < r < 1$.

Summary of fixed points: $x^* = 0 \ \forall r$;

$x^*_\pm = \pm\sqrt{\frac{1}{r} - 1}$ for $0 < r < 1$.

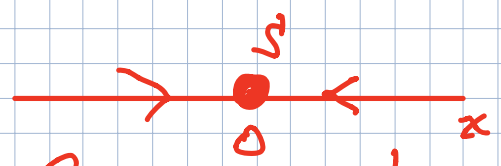
We could have found $x^*_\pm$ graphically by finding intersections of $y = \frac{1}{1+x^2}$ with $y = r$.



From this sketch we see that for $0 < r < 1$, $r - \frac{1}{1+x^2} < 0$ when $x^*_- < x < x^*_+$. It is also < 0 $\forall r < 0$.

Phase Line Diagrams (Vector Fields)

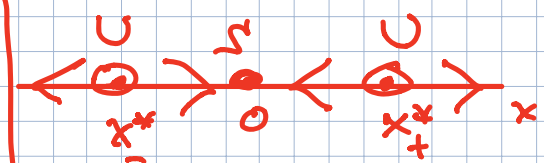
$r<0$



Reason: $r - \frac{1}{1+x^2} < 0$

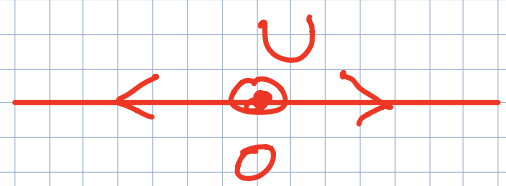
so $x\left(r - \frac{1}{1+x^2}\right) > 0$ when $x<0$ and <0 when $x>0$.

$0<r<1$



Reason: $r - \frac{1}{1+x^2} < 0$ for $x_-^* < x < x_+^*$, so $x\left(r - \frac{1}{1+x^2}\right)$ has the opposite sign from x in this interval, and the same sign as x outside of this interval.

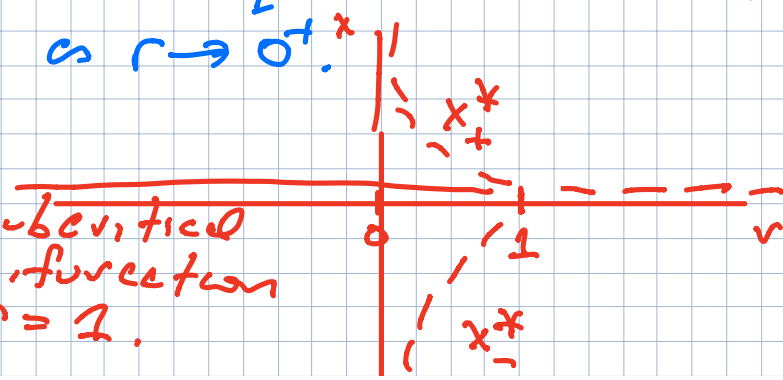
$r>1$



$r - \frac{1}{1+x^2} > 0 \quad \forall x$

so $x\left(r - \frac{1}{1+x^2}\right)$ has the same sign as x.

Bifurcation Diagram: Fixed pt at $x^*=0$ $\forall r$. It is S for $r<1$ and unstable for $r>1$. Fixed points at $x_\pm^* = \pm\sqrt{\frac{1}{r}-1}$ for $0<r<1$ are unstable. Note $x_\pm^* = 0$ for $r=1$ and $x_\pm^* \to \pm\infty$ as $r \to 0^+$.



There is a subcritical pitchfork bifurcation at $r=1$.

Quiz #4: Consider the equation

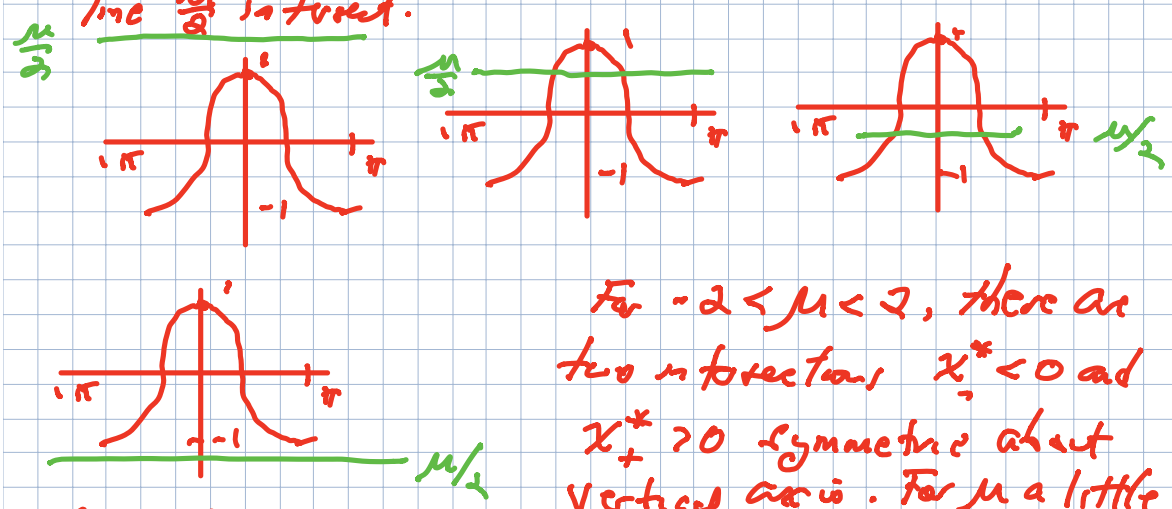
(1) $\dot{\theta} = \mu \sin(\theta) - \sin(2\theta)$.

Draw all qualitatively different phase portraits as μ varies. Find all of the bifurcation points μ and classify each bifurcation.

Using trig identity $\sin 2\theta = 2\sin\theta\cos\theta$, write (1) as

(2) $\dot{\theta} = 2\sin\theta\left(\frac{\mu}{2} - \cos\theta\right)$. $\quad (-\pi < \theta \le \pi)$

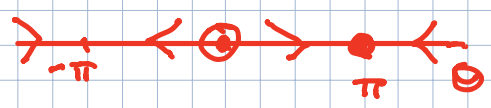
Fixed points are values of θ for which $\sin\theta = 0$ or $\frac{\mu}{2} - \cos\theta = 0$. We always have fixed points at $\theta = 0, \pi$ because $\sin\theta = 0$ there. We have an additional fixed point when θ satisfies (3) $\cos\theta = \frac{\mu}{2}$. Since $-1 \le \cos\theta \le 1$ for all θ, there are sol'ns of (3) for μ satisfying $-2 \le \mu \le 2$. Graphically, these fixed points are where the graph of $\cos(\theta)$ and the horizontal line $\frac{\mu}{2}$ intersect.



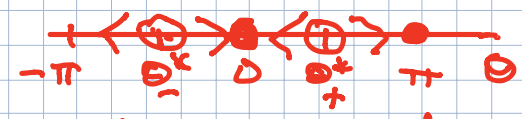
For $-2 < \mu < 2$, there are two intersections $x_-^* < 0$ and $x_+^* > 0$ symmetric about vertical axis. For μ a little less than 2, both fixed points are close to 0. As μ decreases, the fixed points move further apart. As $\mu \to -2^-$, one fixed point approaches $-\pi$ and the other approaches π. For $\mu = 2$, there is one fixed pt at $\theta = 0$ and for $\mu = -2$, one fixed pt at $\theta = \pi$.

For $\mu > 2$, $\frac{\mu}{2} - \cos\theta > 0$ $\forall \theta$, so $\dot\theta$ has same sign as $\sin\theta$

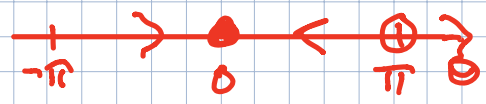
$\mu > 2$



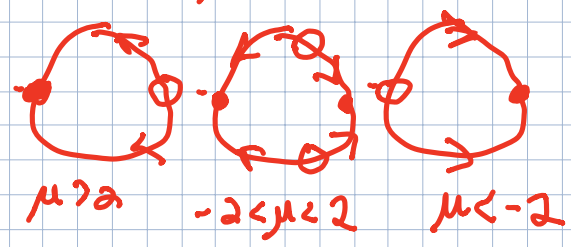
For $2 > \mu > -2$, $\frac{\mu}{2} - \cos\theta < 0$ for $\theta_-^* < \theta < \theta_+^*$ and > 0 otherwise. Using this with the intervals on which $\sin\theta < 0$ or < 0, we get a phase line portrait



For $\mu < -2$, $\frac{\mu}{2} - \cos\theta < 0$ $\forall \theta$, so we have

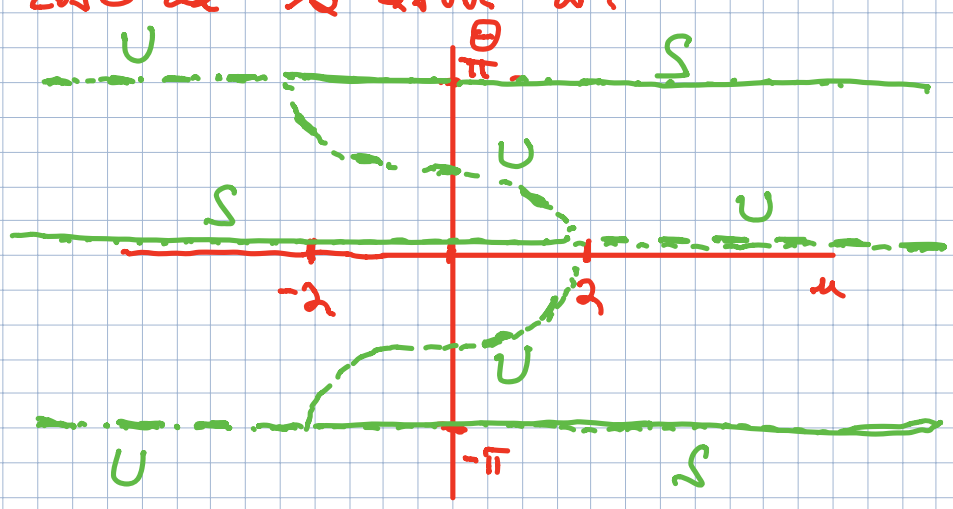


Drawing same into a circle



There is a qualitative change in the phase line for $\mu = -2$ and $\mu = 2$ so these are the bifurcation points. Each bifurcation is a <u>subcritical pitchfork bifurcation</u>.

The bifurcation diagram, reasoned using the graphs of $\cos\theta$ and $\mu/2$ above is:



Alternatively, we can take an analytic approach to determining the bifurcation points.

For $f(\theta) = 2\sin\theta\left(\frac{\mu}{2} - \cos\theta\right)$, a bifurcation occurs for a combination of μ and θ for which

$$f(\theta) = 0 \quad \text{and} \quad f'(\theta) = 0.$$

$$f(\theta) = 2\sin\theta\left(\frac{\mu}{2} - \cos\theta\right)$$

$$f'(\theta) = 2\sin\theta\left(\frac{\mu}{2} - \cos\theta\right)' + 2(\sin\theta)'\left(\frac{\mu}{2} - \cos\theta\right).$$

To have $f(\theta) = 0$, we need either

$$\sin\theta = 0 \qquad \text{or} \qquad \frac{\mu}{2} = \cos\theta$$

If $\sin\theta = 0$, then $f'(\theta) = 2\cos\theta\left(\frac{\mu}{2} - \cos\theta\right)$.

Since $\sin\theta = 0$ for $\theta = 0, \pi$, and $\cos 0 \neq 0$ and $\cos\pi \neq 0$, we'd need

$$\frac{\mu}{2} - \cos(0) = 0 \quad \text{or} \quad \frac{\mu}{2} - \cos(\pi) = 0.$$

These have sol'ns $\mu = 2$, $\theta = 0$ and $\mu = -2$, $\theta = \pi$, as we found graphically.

If $\frac{\mu}{2} - \cos\theta = 0$, then $f'(\theta) = 2\sin\theta\left(\frac{\mu}{2} - \cos\theta\right)'$

$= 2\sin^2\theta$. This $= 0$ for $\theta = 0$ or $\theta = \pi$.

For $\theta = 0$, $\frac{\mu}{2} = \cos\theta$ tells us that $\mu = 2$.

For $\theta = \pi$, $\frac{\mu}{2} = \cos\theta$ tells us that $\mu = -2$, again.

One last note: For $f(\theta) = \mu\sin\theta - \sin 2\theta$, we could try to find fixed points by looking for intersections of $\mu\sin\theta$ and $\sin 2\theta$, but finding such intersections

Graphically is harder than the approach I used above, because both curves have to be drawn fairly precisely to see what is happening.