Math 5470 Spring 2019 Practice Midterm 2

1. Show that the system

$$\begin{aligned} \dot{x} &= x - y - x(x^2+y^2) \\ \dot{y} &= x + y - y(x^2+y^2) \end{aligned}$$

   is equivalent to the system

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

   in polar coordinates. Sketch the phase portrait for this system. Does it have any periodic solution? If so, what is the period?

2. Consider the problem

$$\begin{aligned} \dot{x} &= 3x^2 - 1 - e^{-2y}, \\ \dot{y} &= -2xe^{-2y}. \end{aligned}$$

   Is this a gradient system? How can you tell? If it is a gradient system, find a function $V(x,y)$, such that $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$, where $\mathbf{x} = (x,y)$. If is not a gradient system, determine whether it has a periodic orbit.

3. Consider the system

$$\begin{aligned} \dot{x} &= -x + 2y^3 - 2y^4, \\ \dot{y} &= -x - y + xy. \end{aligned}$$

   Show that this system has no period solutions by constructing a Liapunov function of the form $V(x,y) = x^m + ay^n$ for a suitable choice of m, a, and n.

4. Consider the system:

$$\begin{aligned} \dot{x} &= x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x), \\ \dot{y} &= y(1-4x^2-y^2) + 2x(1+x). \end{aligned}$$

   (a) Show that the origin is an unstable fixed point.

   (b) By considering $\dot{V}$ for $V(x,y) = (1-4x^2-y^2)^2$, show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

5. Consider the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$, where $f(x) = -1$ for $|x| < 1$ and $f(x) = 1$ for $|x| \geq 1$.

   (a) Show that the system is equivalent to the system $\dot{x} = \mu(y - F(x))$, $\dot{y} = -\frac{x}{\mu}$, where $F(x)$ is the piecewise-linear function

$$F(x) = \begin{cases} x+2, & x \leq -1 \\ -x, & |x| < 1 \\ x-2, & x \geq 1. \end{cases} \tag{1}$$

(b) Graph the nullclines.

(c) Show that the system exhibits relaxation oscillations for $\mu >> 1$ and sketch the limit cycle in the (x, y) plane.

(d) Estimate the period of the limit cycle for $\mu >> 1$.