THE BREIMAN CONJECTURE

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*Most of the results in this talk are based on joint work with Péter Kevei and Joel Zinn.

RANDOMLY WEIGHTED SELF–NORMALIZED SUMS

Let $\{Y,\ Y_i : i \geq 1\}$ denote a sequence of i.i.d. random variables, where Y is non–negative with cumulative distribution function [c.d.f.] G.

Now let $\{X,\ X_i : i \geq 1\}$ be a sequence of i.i.d. random variables, independent of $\{Y,\ Y_i : i \geq 1\}$, where X is in the class $\mathcal{X}$ of non-degenerate random variables satisfying for $X \in \mathcal{X}$

$$\mathbb{E}|X| < \infty.$$

For future use, let $\mathcal{X}_0$ denote those $X \in \mathcal{X}$ such that $\mathbb{E}X = 0$. Consider the randomly weighted self–normalized sum

$$\mathbb{T}_n = \sum_{i=1}^{n} X_i Y_i / \sum_{i=1}^{n} Y_i.$$

RANDOMLY SIGNED SELF–NORMALIZED SUMS

Here is a motivating special case.

Let $\{Y,\ Y_i : i \geq 1\}$ and $\{s,\ s_i : i \geq 1\}$ be independent sequences of random variables, where the Y_i's are i.i.d. Y positive and the s_i's are i.i.d. s, where s is the random sign

$$P\{s = 1\} = P\{s = -1\} = 1/2.$$

Consider the randomly signed self–normalized sum,

$$\mathbb{T}_n := \sum_{i=1}^{n} s_i Y_i / \sum_{i=1}^{n} Y_i.$$

THE ARCSINE LAW

The randomly signed self-normalized sum has this interesting motivation. In fair coin tossing, where $+1$ denotes heads and -1 tails, let Y_i be the time between the $(i-1)^{th}$ and i^{th} return to zero of the partial sums $S_1, S_2, \ldots,$ of the coin toss outcomes. Then

$$\left(\mathbb{T}_n + 1\right)/2$$

is the fraction of the time at the n^{th} return to zero that the sums were positive.

In this setup $\left(\mathbb{T}_n + 1\right)/2$ asymptotically has the arcsine law, namely for all $0 \leq x \leq 1$,

$$P\left\{\left(\mathbb{T}_n + 1\right)/2 \leq x\right\} \to \frac{2}{\pi}\arcsin\left(\sqrt{x}\right).$$

DOMAIN OF ATTRACTION

In this talk

$$Y \geq 0 \text{ and } Y \in D\left(\beta\right), 0 \leq \beta < 1,$$

means that for some function L slowly varying at infinity,

$$\overline{G}(y) = y^{-\beta} L(y), \ y > 0,$$

where for any c.d.f. G

$$\overline{G}(y) := P\left\{Y > y\right\}.$$

In the case $0 < \beta < 1$ this is equivalent to Y being in the domain of attraction of a stable law of index β.

BREIMAN RESULT

Among other results, Breiman (1965) proved that $\mathbb{T}_n$ converges in distribution for EVERY $X \in \mathcal{X}$ with at least one limit law being non–degenerate if and only if

$$Y \in D(\beta), \text{ with } 0 \le \beta < 1. \tag{1}$$

BREIMAN CONJECTURE [BC]

At the end of his paper Breiman conjectured that if for *some* $X \in \mathcal{X}$, $\mathbb{T}_n$ converges in distribution to some nondegenerate random variable T, written

$$\mathbb{T}_n \to_d T, \text{ as } n \to \infty, \text{ with } T \text{ nondegenerate,} \tag{2}$$

then (1) holds.

OBSERVATION

By Proposition 2 and Theorem 3 of Breiman (1965), for any $X \in \mathcal{X}$, (1) implies (2), in which case T has a distribution related to the arcsine law. Using this fact, we see that his conjecture can restated to be: for any $X \in \mathcal{X}$,

(1) is equivalent to (2).

A PARTIAL SOLUTION

It has proved to be surprisingly challenging to resolve. Using Karamata's Tauberian theorem, M and Zinn [MZ] (2005) partially verified the Breiman conjecture.

They established that whenever X is nondegenerate and satisfies $\mathbb{E}|X|^p < \infty$ for some $p > 2$, then (1) is equivalent to (2). The $p > 2$ moment condition was imposed in order to conclude that

$$\mathbb{E}\left(\mathbb{T}_n^2\right) \to \mathbb{E}\left(T^2\right) < \infty, \text{ as } n \to \infty.$$

SLIGHT EXTENSION OF MZ

Here is a slight extention of their proof, showing that $\mathbb{E}|X|^2 < \infty$ suffices. Without loss of generality we can assume that $\mathbb{E}X = 0$. An easy calculation gives

$$\mathbb{E}\left(\mathbb{T}_n\right)^2 = var\left(X\right) n\mathbb{E}\left(\frac{Y_1}{Y_1 + \cdots + Y_n}\right)^2.$$

LEMMA

Lemma *Assume that*

$$\mathbb{T}_n \to_d T, \text{ as } n \to \infty,$$

where T *is random variable. Whenever for some* $p \geq 1$, $\mathbb{E}\,|X|^p < \infty$, *then*

$$\mathbb{E}\,|\mathbb{T}_n|^p \to \mathbb{E}\,|T|^p < \infty, \quad \text{as } n \to \infty.$$

IN PARTICULAR

In particular, when $\mathbb{E}|X|^2 < \infty$

$$\mathbb{E}\left(\mathbb{T}_n^2\right) \to \mathbb{E}\left(T^2\right) < \infty, \text{ as } n \to \infty,$$

and thus whenever $\mathbb{E}X = 0$ and T is nondegenerate, for some $0 \leq \beta < 1$,

$$\mathbb{E}\left(\mathbb{T}_n^2\right) = var\left(X\right) n\mathbb{E}\left(\frac{Y_1}{Y_1 + \cdots + Y_n}\right)^2$$

$$\to var\left(X\right)\left(1 - \beta\right).$$

MOMENT RESULT

Arguing as in MZ we get that $\mathbb{E}|X|^2 < \infty$ and (2) suffice for (1), using the following moment result due to Fuks, Ioffe and Teugels (2001), which is proved using Tauberian theorems.

Proposition We have $Y \in D(\beta)$, with $0 \leq \beta < 1$, if and only if

$$n\mathbb{E}\left(\frac{Y_1}{Y_1 + \cdots + Y_n}\right)^2 \to 1 - \beta.$$

CRUCIAL TO THE PROOF

Crucial to the proof of this result was the representation

$$n\mathbb{E}\left(\frac{Y_1}{Y_1+\cdots+Y_n}\right)^2$$

$$= n\int_0^\infty u\varphi''(u)\,(\varphi(u))^{n-1}\,\mathrm{d}u,$$

where $\varphi(u) = \mathbb{E}\exp(-uY_1)$, for $u \geq 0$.

KEVEI-M RESULT

Kevei and M (2015) have further extended the MZ partial solution to the BC. In the following $\phi_X(t)$ denotes the characteristic function of X.

Theorem *Assume that for some* $X \in \mathcal{X}_0$, $1 < \alpha \leq 2$, *positive slowly varying function* L *at zero and* $c > 0$,

$$\frac{-\log\left(\Re\phi_X(t)\right)}{|t|^{\alpha} L\left(|t|\right)} \to c, \text{ as } t \to 0. \qquad (3)$$

Whenever (2) holds then $Y \in D(\beta)$ *for some* $\beta \in [0, 1)$.

COROLLARY

Let $\mathcal{F}$ denote the class of random variables that satisfy the conditions of the theorem. Applying our theorem in combination with Proposition 2 and Theorem 3 of Breiman (1965) we get the following corollary.

Corollary *Whenever* $X - \mathbb{E}X \in \mathcal{F}$*, (1) is equivalent to (2).*

IMPORTANT OBSERVATIONS

It can be inferred from Theorem 8.1.10 of Bingham, Goldie, and Teugels (1987) that for $X \in \mathcal{X}_0$, (3) holds for some $1 < \alpha < 2$, positive slowly varying function L at zero and $c > 0$ if and only if X satisfies

$$\mathbb{P}\{|X| > x\} \sim L(1/x)x^{-\alpha}c\Gamma(\alpha)\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right).$$

Note that a random variable $X \in \mathcal{X}_0$ in the domain of attraction of a stable law of index $1 < \alpha < 2$ satisfies (3).

Also a random variable $X \in \mathcal{X}_0$ with variance $0 < \sigma^2 < \infty$ fulfills (3) with $\alpha = 2$, $L = 1$ and $c = \sigma^2/2$. This means that the Kevei-M theorem contains the MZ result.

PROPOSITION 1

The theorem is a consequence of the two propositions that follow. First we need more notation. For any $\alpha \in (1, 2]$ define for $n \geq 1$

$$S_n(\alpha) = \frac{\sum_{i=1}^{n} Y_i^{\alpha}}{(\sum_{i=1}^{n} Y_i)^{\alpha}}.$$

Proposition 1 *Assume that the assumptions of the theorem hold. Then for some* $0 < \gamma \leq 1$

$$\mathbb{E} S_n(\alpha) \to \gamma, \text{ as } n \to \infty. \tag{4}$$

PROPOSITION 2

The next proposition is interesting in its own right. It is an extension of Theorem 5.3 of Fuchs, Joffe and Teugels (2002), where $\alpha = 2$ (see also Proposition 3 of MZ).

Proposition 2 *If (4) holds with some* $\gamma \in (0, 1]$ *then* $Y \in D(\beta)$*, for some* $\beta \in [0, 1)$*, where* $-\beta \in (-1, 0]$ *is the unique solution of*

$$\frac{\Gamma(\alpha - 1)\Gamma(1 - \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

In particular, $Y \in D(0)$ *for* $\gamma = 1$*.*
Conversely, if $G \in D(\beta)$, $0 \leq \beta < 1$*, then (4) holds with*

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)}.$$

KEY RESULT USED IN PROOF

Set for each $n \geq 1$, for $i = 1, \ldots, n$

$$R_{i,n} = Y_i / \sum_{l=1}^{n} Y_l.$$

Consider the sequence of strictly decreasing continuous functions $\{\varphi_n\}_{n \geq 1}$ on $[1, \infty)$ defined for $y \in [1, \infty)$ by

$$\varphi_n(y) = \mathbb{E}\left(\sum_{i=1}^{n} R_{i,n}^{y}\right).$$

LEMMA

Note that each function φ_n satisfies $\varphi_n(1) = 1$. By a diagonal selection procedure for each subsequence of $\{n\}_{n\geq 1}$ there is a further subsequence $\{n_k\}_{k\geq 1}$ and a right continuous non-increasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ.

Lemma *Each such function ψ is continuous on $(1, \infty)$.*

KEY STEP USED IN PROOF

One can show that

$$\mathbb{E}\frac{\sum_{i=1}^n Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} =$$

$$\frac{n}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}\mathbb{E}\left(e^{-uY_1}Y_1^\alpha\right)(\mathbb{E}e^{-uY_1})^{n-1}\mathrm{d}u,$$

which if it converges to $\gamma \in (0,1]$, then

$$t^{\alpha-1}\frac{\int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u}{\Gamma(\alpha)\int_0^\infty \overline{G}(u)e^{-ut}\mathrm{d}u} \to \gamma, \text{ as } t \searrow 0. \tag{5}$$

APPLICATION OF DRASIN-SHEA THEOREM

An application of a version of the Drasin-Shea theorem shows that (5) implies $G \in D(\beta)$, where $0 \le \beta < 1$ satisfies

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)}.$$

MELLIN TRANSFORM -CONVOLUTION

Mellin transform Given a measurable kernel $k : (0, \infty) \to \mathbb{R}$, its Mellin transform is

$$\widetilde{k}\,(z) = \int_{(0,\infty)} t^{-z-1} k\,(t)\,dt$$

for $z \in \mathbb{C}$, such that $\widetilde{k}\,(z)$ is finite.

Mellin convolution For suitable functions f and $g : (0, \infty) \to \mathbb{R}$,

$$f \overset{M}{*} g\,(x) = \int_{(0,\infty)} t^{-1} f\,(x/t)\,g\,(t)\,dt.$$

BOUNDED DECREASE

The **lower Matuszewska index** $\beta(f)$ is the supremum of those β for which there exists a constant $d = d(\beta) > 0$ such that for each $\Lambda > 1$, as $x \to \infty$, uniformly in $\lambda \in [1, \Lambda]$,

$$f(\lambda x) / f(x) \geq d \{1 + o(1)\} \lambda^{\beta}.$$

Bounded decrease [BD] A positive function f has bounded decrease if $\beta(f) > -\infty$.

DRASHIN-SHEA THEOREM

Theorem Let k be a non-negative kernel whose Mellin transform $\widetilde{k}$ has maximal convergence strip $a < \Re z < b$, where $a < 0$. If a is finite, assume that $\widetilde{k}\,(a+) = \infty$, if b is finite assume $\widetilde{k}\,(b-) = \infty$. Let f be a non-negative, locally bounded on $[0, \infty)$. Assume that $f \in BD$. If

$$k \overset{M}{*} f\,(x)\,/f\,(x) \to c, \text{ as } x \to \infty,$$

holds, then $c = \widetilde{k}\,(\rho)$ for some $\rho \in (a, b)$, and $f \in R_\rho$, meaning that f is regularly varying at infinity with index ρ.

OUR KERNEL

The particular kernel that we use is

$$k(u) = \begin{cases} (u-1)^{\alpha-2} u^{-\alpha+1}, & u > 1, \\ 0, & 0 < u \leq 1. \end{cases}$$

Its *Mellin-transform* is

$$\begin{aligned} \widetilde{k}\,(z) &= \int_1^\infty (u-1)^{\alpha-2}\, u^{-\alpha-z} \mathrm{d}u \\ &= \frac{\Gamma\,(\alpha-1)\,\Gamma\,(1+z)}{\Gamma\,(\alpha-z)}, \end{aligned}$$

which is convergent for $z > -1$.

FURTHER RESULTS

Kevei and M (2012) studied the asymptotic distribution of randomly weighted sums $\mathbb{T}_n$ along subsequences $\{n'\}$ of $\{n\}$.

Our basic result is the following extension of Proposition 3 of Breiman (1965), who proved it in the case $Y \in D(\beta)$, with $0 < \beta < 1$, and for the full sequence $\{n\}$.

Assume that $\mathbb{E}|X| < \infty$ and for a subsequence $\{n'\}$ there exists a sequence of positive norming constants $a_{n'}$ such that

$$\frac{1}{a_{n'}} \sum_{i=1}^{n'} Y_i \xrightarrow{\mathrm{D}} W_2, \text{ as } n' \to \infty,$$

where W_2 is an $id(0, b, \Lambda)$ infinitely divisible random variable.

$id(0, b, \Lambda)$ RANDOM VARIABLE

This means that W_2 is an infinitely divisible random variable taking values in $[0, \infty)$ with characteristic exponent

$$\log \mathbb{E} \mathrm{e}^{\mathrm{i}uW_2}$$

$$= \mathrm{i}ub + \int \left(\mathrm{e}^{\mathrm{i}ux} - 1 - \mathrm{i}uxI(|x| \leq 1) \right) \Lambda(\mathrm{d}x),$$

$b \geq \int_0^1 x\Lambda(dx)$ and Lévy measure Λ concentrated on $(0, \infty)$.

CONCLUSION

Then along the same subsequence $\{n'\}$, as $n' \to \infty$,

$$\left(\frac{\sum_{i=1}^{n'} X_i Y_i}{a_{n'}}, \frac{\sum_{i=1}^{n'} Y_i}{a_{n'}}\right) \xrightarrow{\mathrm{D}} (W_1, W_2),$$

where

$$(W_1, W_2) \stackrel{\mathrm{D}}{=} (a_1 + U, a_2 + V)$$

with $(a_1, a_2) =$

$$\left(\left(b - \int_0^1 x\Lambda(\mathrm{d}x)\right)\mathbb{E}X, b - \int_0^1 x\Lambda(\mathrm{d}x)\right)$$

and $\mathbb{E}\mathrm{e}^{\mathrm{i}(\theta_1 U + \theta_2 V)} =$

$$\exp\left\{\int_0^\infty \int_{-\infty}^\infty \left(\mathrm{e}^{\mathrm{i}(\theta_1 x + \theta_2 y)} - 1\right) \mathrm{d}F\left(x/y\right)\Lambda\left(\mathrm{d}y\right)\right\}$$

$$=: \exp\left\{\phi\left(\theta_1, \theta_2\right)\right\}.$$

IMPLICATION

We see that whenever

$$\frac{1}{a_{n'}}\sum_{i=1}^{n'} Y_i \xrightarrow{\text{D}} W_2, \text{ as } n' \to \infty,$$

and $P\{W_2 > 0\} = 1$, then as $n' \to \infty$

$$\mathbb{T}_{n'} = \sum_{i=1}^{n'} X_i Y_i / \sum_{i=1}^{n'} Y_i. \xrightarrow{\text{D}} W_1/W_2.$$

In particular this happens when Y is in the centered Feller class, since in this case necessarily W_2 has a Lebesgue density.

FELLER CLASS

ξ is in the Feller class if for the sequence of partial sums $\{S_n\}_{n\geq 1}$ of i.i.d. ξ there exist norming and centering constants $B(n) > 0$, $A(n)$ such that every subsequence of $\{n_k\}$ of $\{n\}$ contains a further subsequence $n_{k'} \to \infty$ with

$$\frac{S_{n_{k'}} - A(n_{k'})}{B(n_{k'})} \xrightarrow{\text{D}} W,$$

where W is a finite nondegenerate rv depending on the subsequence $n_{k'}$, which by a result of Pruitt (1983) has a Lebesgue density.

We shall write this as "$\xi \in FC$".

CENTERED FELLER CLASSES

If the centering function $A(n)$ can chosen to be identically equal to zero, we shall say:

ξ is in the centered Feller class, written "$\xi \in FC_0$".

THE CASE $Y \in FC_0$

Whenever $Y \in FC_0$ one can show that $a_1 = a_2 = 0$ and thus all of the subsequential distributional limits of $\mathbb{T}_n$ are of the form U/V with $P\{V > 0\} = 1$.

Moreover, by using a result of Griffin (1986) it can be shown that these distributional limits have Lebesgue densities on $\mathbb{R}$.

GOING THE OTHER WAY

Going the other way, Kevei and M (2013) show that if all of the subsequential laws of $\mathbb{T}_n$ are continuous for any choice of $X \in \mathcal{X}$, then necessarily $Y \in FC_0$.

In particular, if $Y \in FC$, but $Y \notin FC_0$ then there exists a subsequence $\{n'\}$ such that

$$\sum_{i=1}^{n'} X_i Y_i / \sum_{i=1}^{n'} Y_i \xrightarrow{\mathrm{P}} \mathbb{E}X, \text{ as } n' \to \infty.$$

ASSOCIATED BIVARIATE LÉVY PROCESS

Under the assumptions of our basic result we can readily prove that for any $t > 0$, as $n' \to \infty$,

$$\left(\frac{\sum_{1\le i\le n't} X_i Y_i}{a_{n'}}, \frac{\sum_{1\le i\le n't} Y_i}{a_{n'}}\right) \xrightarrow{\mathrm{D}} (a_1 t + U_t, a_2 t + V_t),$$

where (U_t, V_t), $t \geq 0$, is the bivariate Lévy process with characteristic function

$$\mathbb{E}\mathrm{e}^{\mathrm{i}(\theta_1 U_t + \theta_2 V_t)} =: \exp\left\{t\phi\left(\theta_1, \theta_2\right)\right\}.$$

ASYMPTOTIC DISTRIBUTION OF RATIO

Kevei and M (2013) have characterized when under regularity conditions that the ratio

$$T_t := U_t/V_t \xrightarrow{\mathrm{D}} T.$$

converges in distribution to a nondegenerate random variable T as $t \to \infty$ or $t \searrow 0$.

KEVEI-M RESULT

They obtained the following analog for T_t of the MZ result:

Suppose X is non-degenerate and satisfies $\mathbb{E}|X|^p < \infty$ for some $p > 2$, then the ratio as $t \to \infty$ $(t \searrow 0)$

$$T_t := U_t / V_t \stackrel{\mathrm{D}}{\to} T, \text{ nondegenerate,}$$

(in the case $t \downarrow 0$ we assume $\overline{\Lambda}(0+) = \infty$.) if and only if for some $0 \le \beta < 1$,

$\overline{\Lambda}(x)$ is regularly varying at infinity (zero) with index $-\beta$.

LEVY PROCESS FELLER CLASS

A Lévy process Y_t is said to be in the *Feller class* at infinity if there exists a norming function $B(t)$ and a centering function $A(t)$ such that for each sequence $t_k \to \infty$ there exists a subsequence $t'_k \to \infty$ such that

$$\left(Y_{t'_k} - A(t'_k)\right)/B(t'_k) \xrightarrow{\mathrm{D}} W, \quad \text{as } k \to \infty,$$

where W is a nondegenerate random variable.

For the definition of *Feller class* at zero replace $t_k \to \infty$ and $t'_k \to \infty$, by $t_k \searrow 0$ and $t'_k \searrow 0$, respectively.

LEVY PROCESS CENTERED FELLER CLASS

The Lévy process Y_t belongs to the *centered Feller class* at infinity if it is in the Feller class at infinity and the centering function $A(t)$ can be chosen to be identically zero.

For the definitions of *centered Feller class* at zero replace $t_k \to \infty$ and $t'_k \to \infty$, by $t_k \searrow 0$ and $t'_k \searrow 0$, respectively.

CONTINUOUS LIMITS

Theorem *All subsequential distributional limits of* U_t/V_t*, as* $t \searrow 0$*, (as* $t \to \infty$*) are continuous for any cdf* F *in the class* $\mathcal{X}$*, if and only if* V_t *is in the centered Feller class at 0* (∞)*.*

REPRESENTATION

Now let $\{X_s\}_{s\geq 0}$ be a class of i.i.d. F random variables independent of the V_t process. It turns out that for each $t \geq 0$ the bivariate process

$$(U_t, V_t) \stackrel{\mathrm{D}}{=} \left(\sum_{0\leq s\leq t} X_s \Delta V_s, \sum_{0\leq s\leq t} \Delta V_s \right), \quad (6)$$

where $\Delta V_s = V_s - V_{s-}$.

Notice that in the representation (6) each jump of V_t is weighted by an independent X_t so that U_t can be viewed as a randomly weighted Lévy process.

A PICTURE OF THIS PROCESS

Here is a graphic way to picture this bivariate process. Consider ΔV_s as the intensity of a random shock to a system at time $s > 0$ and $X_s \Delta V_s$ as the cost of repairing the damage that it causes.

Then V_t, U_t and U_t/V_t represent, respectively, up to time t, the total intensity of the shocks, the total cost of repair and the average cost of repair with respect to shock intensity.

A MOTIVATING EXAMPLE

Let ΔV_s represent a measure of the intensity of a tornado that comes down in a Midwestern American state at time s during tornado season and X_s the cost of the repair of the damage per intensity that it causes.

Note that X_s is a random variable that depends on where the tornado hits the ground, say a large city, a medium size town, a village, an open field, etc.

It is assumed that a tornado is equally likely to strike anywhere in the state.