Poincaré Duality, Bakry–Émery Estimates and Isoperimetry on Fractals

Daniel J. Kelleher

Golomb Visiting Assistant Professor
Department of Mathematics
Purdue University

Joint work with Fabrice Baudoin

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Bakry–Émery Gradient Estimates

If Γ is an appropriate notion of gradient, and P_t is an associated heat kernel, the Bakry–Émery Gradient estimates

$$\sqrt{\Gamma(P_t f)} \leq P_t \sqrt{\Gamma(f)}.$$

Can be used to establish

1. Riesz-Transform Bounds
   (Coulhon and Duong et al.)
2. Isoperimetric inequalities
   (e.g. Baudoin–Bonnefont)
3. Wasserstein Control
   (Kuwada Duality)

Generalizations of Curvature

The Bakry–Émery estimate can be thought of as a curvature condition.

In the appropriate settings it is equivalent to

1. Curvature Dimension Inequalities of Bakry-Émery.
2. Ricci Curvature Lower bounds of Lott–Villani.

Question Can we find a situation which supports a Bakry–Émery gradient estimate, but neither of the above?

Setting

We have the classical Dirichlet energy, on $\mathbb{R}^n$

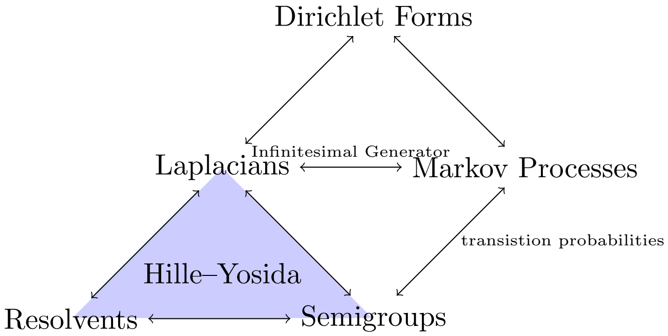
$$\mathbf{D}(f) = \int |\nabla f|^2 \, dx \quad f \in H^1(\mathbb{R}^n)$$

by $H^1(\mathbb{R}^n)$, can be either seen as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^1}^2 = \int |f|^2 + |\nabla f|^2 \, dx.$$

Or you can think of $H^1(\mathbb{R}^n)$ as the set of functions $f \in L^2(\mathbb{R})$ such that $|\nabla f| \in L^2(\mathbb{R})$,

Setting


Dirichlet Forms
Laplacians
Infinitesimal Generator
Markov Processes
transistion probabilities
Hille–Yosida
Resolvents
Semigroups

Energy Measures $\nu_{f,g}$ such that

$$\int \phi \, d\nu_{f,g} = \mathcal{E}(f\phi, g) + \mathcal{E}(g\phi, f) - \mathcal{E}(\phi, fg).$$

$\mathcal{E}$ admits a Carré du Champ/μ is energy dominant

$$\mu \ll \nu_{f,g} \text{ for all } f \text{ and define } \Gamma_\mu(f,g) = \frac{d\nu_{f,g}}{d\mu}$$

Classical Case: $\Gamma(f,g) = \nabla f \cdot \nabla g$

Setting

- (X, d) is a locally compact Hausdorff space
- μ Borel regular measure with volume doubling, i.e. there is some constant C_{vol}

$$C_{vol}\mu(B_{2r}(x)) \leq \mu(B_r(x)) \quad \text{and} \quad \mu(B_1(x)) \geq c_{vol}$$

- $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a local regular Dirichlet form with heat semigroup P_t.
- Energy Measures $\nu_{f,g}$ such that

$$2\int \phi \, d\nu_{f,g} = \mathcal{E}(f\phi, g) + \mathcal{E}(g\phi, f) - \mathcal{E}(\phi, fg).$$

- $\mathcal{E}$ admits a Carré du Champ/μ is energy dominant

$$\mu \ll \nu_{f,g} \text{ for all } f \text{ and define } \Gamma_\mu(f, g) = \frac{d\nu_{f,g}}{d\mu}$$

- Poincaré inequality

$$C \int_{B_r(x)} \left| f - \overline{f}_{B_r(x)} \right| \, d\mu \leq \nu_f(B_{C_P r}(x))$$

General Results

Reisz Transform: $f \mapsto \Gamma_\mu(\Delta^{-1/2} f)$.

Theorem

If we have

- *Locally compact Hausdorff metric space* (X, d).
- *Upper and lower volume Doubling measure* μ.
- *Dirichlet form* $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ *which admits a Carré du Champ.*

Which Satisfy

- *Poincaré Inequality*
- *Bakry–Émery inequality*

Then the **Riesz Transform** *is bounded for* $p \geq 1$, *i.e.*

$$\left\| \Gamma_\mu(f, f)^{1/2} \right\|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p$$

Perimeters and Bounded Variation

We say f is **bounded variation**, and write $f \in BV$, if

$$\lim_{t \to 0} \int \sqrt{\Gamma(P_t f)}\, d\mu < \infty$$

and define $\mathrm{Var}(f) = \lim_{t \to 0} \int \sqrt{\Gamma(P_t f)}\, d\mu$.

If $\mathbf{1}_E \in BV$, we then the **perimeter** is called $\mathrm{P}(E) = \mathrm{Var}(\mathbf{1}_E)$.

E is called a **Caccioppoli set** if $\mathbf{1}_E \in BV$.

Isoperimetric Inequalities

Theorem (Baudoin-K.)

If we have

- *Locally compact Hausdorff metric space* (X, d).
- *Upper and lower volume Doubling measure* μ.
- *Dirichlet form* $(\mathcal{E}, \operatorname{dom} \mathcal{E})$.

Which Satisfy

- *Poincaré Inequality and Bakry–Émery Inequality*

Then **Isoperimetric Inequality** *there exists* Q *and* C_{iso} *such that*

$$\mu(E)^{1-1/Q} \leq C_{iso} \, \mathrm{P}(E).$$

and **Gaussian Isoperimetric Inequality**

$$C\mu(E)\sqrt{\ln\left(1/\mu(E)\right)} \leq \mathrm{Per}(E).$$

Cheeger's Constant and Spectral gaps

Cheeger Constant

$$h = \inf \frac{P(E)}{\mu(E)}$$

Theorem

The spectral gap

$$\lambda_1 \leq \frac{2h^2}{(1 - e^{-1})^2}$$

Gaussian Isoperimetric constant

If

$$k = \inf \frac{P(E)}{\mu(E)\sqrt{-\mu(E)}}$$

for $\mu(E) \leq 1/2$.

Theorem

If ρ_0 is the log-Sobolev constant, then

$$\rho_0 \leq 512k^2.$$

Here we mean that ρ_0 is optimal constant such that

$$\int f^2 \ln f^2 \, d\mu - \int f^2 \, d\mu \ln\left(\int f^2 \, d\mu\right) \leq \frac{1}{\rho_0}\mathcal{E}(f).$$

Kuwada Duality

Let

$$W_p(\nu_1, \nu_2) = \inf_{\pi} \left(\int d(x,y)^p \; \pi(dx, dy) \right)^{1/p}$$

be the p-Wasserstein Distance between two probability measures on a metric measure space (X, d).

Then there is a dual form of the Bakry–Émery inequality called p-Wasserstein control:

$$W_p(P_t^* \nu_1, P_t^* \nu_2) \leq e^{-kt} W_p(\nu_1, \nu_2).$$

Where

$$\int f dP_t^* \nu = \int P_t f \; d\nu.$$

Theorem (Kuwada)

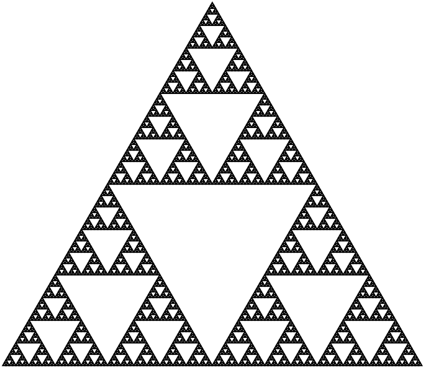
p-Wasserstein control:

$$W_p(P_t^*\nu_1, P_t^*)\nu_2 \leq e^{-kt} W_p(\nu_1, \nu_2).$$

is Equivalent to

$$\sqrt{\Gamma(P_t f)} \leq e^{-kt}(P_t(\Gamma(f))^{p/2})^{1/p}$$

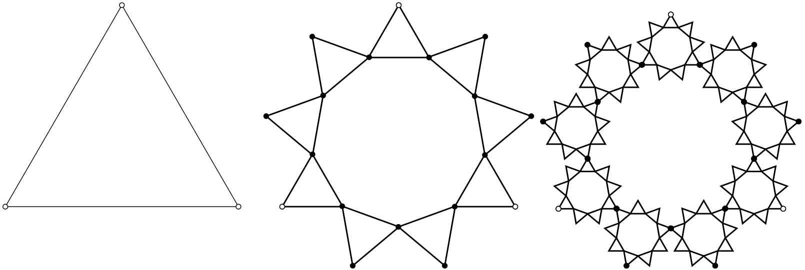
Poincare Duality On Fractals



Goals

- To Classify the differential forms on one dimensional Dirichlet spaces, particularly on the Sierpinski Gasket.
- Relate the heat equation on differential forms to that on scalars.

P.C.F Self-similar structures



Approximating graphs G_k with vertex sets

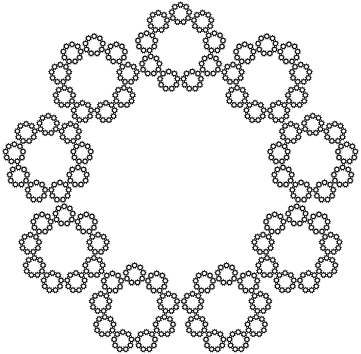
$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset X.$$

We have "compatible" graph energies $\mathcal{E}_k$ on G_n,

$$\mathcal{E}_j(f) = \inf \left\{ \mathcal{E}_k(g) \mid g|_{V_j} = f \right\}.$$

Because of the symmetry $\mathcal{E}_j$ is the graph energy on V_j scaled by a constant.

P.C.F. Self-similar structures

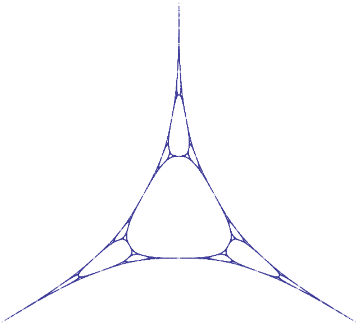


Then there is a self-similar form

$$\mathcal{E}(f) = \lim \mathcal{E}_j(f|_{V_j})$$

Where $\text{dom}_{\mathcal{E}} = \{f : K \to \mathbb{R} \mid \mathcal{E}(f) < \infty\}$.

Harmonic Energy Measures



It is possible to define Harmonic functions h with boundary at the corners, by solving the Dirichlet problem on the graphs and taking the limit.
We shall consider the reference measure $\mu = \nu_h$.

Differential forms on fractals

Idea: to deal with these problems by developing a differential geometry for Dirichlet spaces and hence fractals

based on
Differential forms on the Sierpinski gasket and other papers by Cipriani–Sauvageot

Derivations and Dirichlet forms on fractals
by Ionescu–Rogers–Teplyaev, JFA 2012

Vector analysis on Dirichlet Spaces
by Hinz–Röckner–Teplyaev, SPA 2013

Differential forms on Dirichlet spaces

Let X be a locally compact second countable Hausdorff space and m be a Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X, m)$.

Write $\mathcal{C} := C_0(X) \cap \mathcal{F}$.

The space $\mathcal{C}$ is a normed space with

$$\|f\|_{\mathcal{C}} := \mathcal{E}_1(f)^{1/2} + \sup_{x \in X} |f(x)|.$$

Differential forms on Dirichlet spaces

We equip the space $\mathcal{C} \otimes \mathcal{C}$ with a bilinear form, determined by

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \int_X bd \, d\Gamma(a, c).$$

This bilinear form is nonnegative definite, hence it defines a seminorm on $\mathcal{C} \otimes \mathcal{C}$.

$\mathcal{H}$: the Hilbert space obtained by factoring out zero seminorm elements and completing.

Differential forms on Dirichlet spaces

In the classical setting, this norm

$$\int |b|^2 |\nabla a|^2 \; d\mu$$

Where μ is Lebesgue measure in the appropriate dimension.

And, any simple tensor $a \otimes b = \sum_{i=1}^{d} x^i \otimes b \frac{\partial a}{\partial x^i}$.

Think of $x^i \otimes \mathbf{1}$ as dx^i,

Differential forms on Dirichlet spaces

We call $\mathcal{H}$ the *space of differential* 1-*forms* associated with $(\mathcal{E}, \mathcal{F})$.

The space $\mathcal{H}$ can be made into a $\mathcal{C}$-$\mathcal{C}$-bimodule by setting

$$a(b \otimes c) := (ab) \otimes c - a \otimes (bc) \quad \text{and} \quad (b \otimes c)d := b \otimes (cd)$$

and extending linearly.

$\mathcal{C}$ acts on both sides by uniformly bounded operators.

Differential on Dirichlet spaces

we can introduce a derivation operator by defining $\partial : \mathcal{C} \to \mathcal{H}$ by $\partial a := a \otimes \mathbf{1}$.

$\|\partial a\|^2 \leq 2\mathcal{E}(a)$ and the **Leibniz rule** holds,

$$\partial(ab) = a\partial b + b\partial a, \quad a, b \in \mathcal{C}.$$

Co-Differential on Dirichlet spaces

The operator ∂ extends to a closed unbounded linear operator from $L_2(X, m)$ into $\mathcal{H}$ with domain $\mathcal{F}$.

Let ∂^* denote its adjoint, such that

$$\langle \partial^* \omega, g \rangle_{L^2} = \langle \omega, \partial g \rangle_{\mathcal{H}} \tag{1}$$

Let $\mathcal{C}^*$ be the dual space of the normed space $\mathcal{C}$. Then ∂^* defines a bounded linear operator from $\mathcal{H}$ into $\mathcal{C}^*$.

In this talk we shall consider $\partial^* : \mathcal{H} \to L^2(X)$ by restricting to the domain

$$\operatorname{dom} \partial^* = \left\{ \eta \in \mathcal{H} \mid \exists f \in L^2(X) \; with \; \partial^* \eta(\phi) = \langle f, \phi \rangle_{L^2} \right\}$$

PDE on fractals

We can think of ∂ as something like a gradient or an exterior derivative.

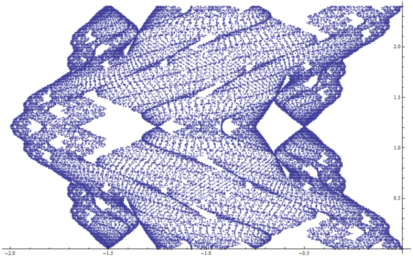
And think of ∂^* as div or as the co-differential.

This allows for a lot of new differential equations to but represented on fractals

For instance, we now have a divergence form

$$\partial^* a(\partial u) = 0$$

Magnetic Schrödinger operators



Classically

$$i\frac{\partial u}{\partial t} = (-i\nabla - A)^2 u + Vu$$

becomes

$$i\frac{\partial u}{\partial t} = (-i\partial - a)^*(-i\partial - a)u + Vu$$

Where $a \in \mathcal{H}$ and $V \in L_\infty(X, m)$.

Definition of Poincare Duality

A result of *Hinz–Röckner–Teplyaev* shows that (with some technical conditions) there is a "fibrewise" inner product and norm on $\mathcal{H}$. Call the fibres $\mathcal{H}_x$ and the inner product $\langle ., . \rangle_{\mathcal{H},x}$.

Note

$$\langle \partial f, \partial g \rangle_{\mathcal{H},x} = \Gamma_\mu(f, g)(x)$$

almost everywhere.

Definition of Poincare Duality

Theorem (Baudoin–K.)

In the above situation, chose $\omega \in \mathcal{H}$ *such that* $\|\omega\|_{\mathcal{H},x} = 1$ μ*-a.e. then* $\star L^2(X,\mu) \to \mathcal{H}$ *defined by*

$$\star f = \omega \cdot f$$

is a isometry both globally and fiberwise with inverse

$$\star\eta(x) = \langle \omega, \eta \rangle_{\mathcal{H},x} .$$

In particular $L^2(X,\mu) \cong \mathcal{H}$ *as Hilbert spaces.*

Proof Hino index 1 implies that $\dim \mathcal{H}_x = 1$ almost everywhere.

Laplacian on Differential Forms

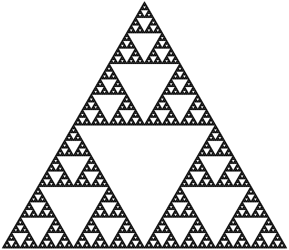
Consider

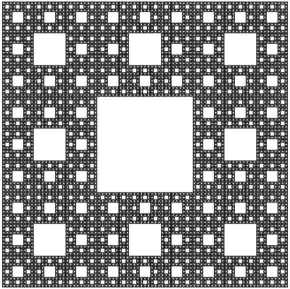
$$\vec{\Delta} = \partial \partial^*$$

with domain

$$\operatorname{dom} \vec{\Delta} = \{\omega \in \mathcal{H} \mid \partial^* \omega \in \operatorname{dom} \partial\}.$$

Hodge Decomposition





Hinz–Teplyaev: When restricted to topologically 1-dimensional fractals, there is a Hodge decomposition with

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

where

$$\mathcal{H}^0 = \operatorname{Im} \partial \quad \text{are Exact Forms}$$

and

$$\mathcal{H}^1 = \ker \partial^* \quad \text{are Harmonic Forms}$$

Product rule for ∂^*

The co-differential has the following product rule

$$\partial^*(\eta \cdot f) = \langle \partial f, \eta \rangle_{\mathcal{H}_x} + f\partial^*\eta.$$

Thus if $\omega \in \mathcal{H}^1$ is harmonic, then the second term on the right disappears and we get.

$$\partial^* \star f = \star\partial f.$$

Note: It is **not true that**

$$\star\partial^*\eta = \partial \star \eta$$

Classification of Differential forms

Theorem (Baudoin–K.)

Consider the self-similar energy form $\mathcal{E}$ on SG, with respect to a borel measure μ,

1. $\mu = \nu_h$ *is the energy measure associated to the harmonic* h *with boundary* V_0.
2. $\star$ *is the Hodge Star with respect to* ∂h.
3. Δ_0 *is the Dirichlet Laplacian with boundary* V_0.

Then $\vec{\Delta}$ *restricted to exact forms* $\mathcal{H}^0$ *is equal to* $-\star\Delta_0\star$ *as operators.*

If $\Delta_\mu = \partial^*\partial$ is the generator of $\mathcal{E}$ with respect to μ, this implies that

$$\operatorname{dom}\Delta_\mu = \{f \in \operatorname{dom}\mathcal{E} \mid \star\partial f = \Gamma(f,h) \in \operatorname{dom}_0\mathcal{E}\}$$

Energy measures can be extended to elements of $\mathcal{H}$ by

$$\int \phi \, d\nu_\omega := \langle \omega \cdot \phi, \omega \rangle_{\mathcal{H}} .$$

Theorem (Baudoin–K.)

Consider the self-similar energy form $\mathcal{E}$ on SG, with respect to a borel measure μ,

1. *$\mu = \nu_\omega$ is the energy measure associated to the harmonic form omega $\in \mathcal{H}^1$.*
2. *$\star$ is the Hodge Star with respect to ω.*
3. *Δ_ω is the generator of $\mathcal{E}$.*

Then $\vec{\Delta}$ restricted to exact forms $\mathcal{H}^0$ is equal to $\star\Delta\star$ as operators.

Bakry–Émery Inequality on the Sierpinski Gasket

Theorem (Baudoin–K.)

In either of the settings of the above theorems, the Bakry–Émery inequality is satisfied.

That is if μ is either ν_h for some harmonic function h, or ν_ω for some harmonic form ω, then

$$\sqrt{\Gamma_\mu(e^{-t\Delta_\mu}f)} \leq e^{-t\Delta_\mu}\sqrt{\Gamma_\mu(f)}.$$

Proof of Bakry–Émery inequality

Idea:

$$e^{t\vec{\Delta}}\partial = \partial e^{t\Delta}$$

Then because $\vec{\Delta} = \star\Delta\star$

$$\star e^{-t\Delta} \star \partial = \partial e^{-t\Delta}.$$

Thus

$$\left|e^{t\Delta} \star \partial f(x)\right| = \left\|\partial e^{t\Delta} f\right\|_{\mathcal{H},x} = \sqrt{\Gamma(e^{t\Delta} f)(x)}.$$

The Inequality follows from the fact that

$$\left|e^{t\Delta} \star \partial f(x)\right| \leq e^{t\Delta} \left|\star\partial f\right| = e^{t\Delta}\sqrt{\Gamma(f)}.$$

Reisz Transform and Differential Forms

The Riesz transform in this case can be interpreted as

$$\mathbf{R} = \star \partial \Delta^{1/2}$$

Theorem (Baudoin–K.)

Let $p > 1$ then for every $f \in L^2(X)$ with $\int f \, d\mu = 0$, then

$$\|\mathbf{R} f\|_p \leq 2(p^* - 1)\, \|f\|_p$$

where $p^* = \max\{p, p/(p-1)\}$

Reisz Transform: Gundy Representation

Let X_t be the diffusion generated by Δ and started with distribution μ. Let B_t be Brownian motion of $\mathbb{R}$. Define, for $f \in L^2(X)$

$$Qf(x,y) = e^{-y\sqrt{-\Delta}} f(x)$$

Then $M_t^f = Q(X_t, B_t)$ is a Martingale, and we have the following Gundy Representation (see also to Bañuelos–Wang)

Lemma

For $f \in L^2(X)$, $\int f = 0$ then

$$\mathbf{R}f(x) = -2 \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(\int_0^{\tau_0} \star \partial Qf(X_s, B_s)\, dB_s \middle| X_{\tau_0} = x \right)$$

where $\tau_0 = \inf \{t : B_t = 0\}$

Reisz Transform: Martingale Subordination

$$\mathbf{R}f(x) = -2 \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(\int_0^{\tau_0} \star \partial Q f(X_s, B_s) \, dB_s \Big| X_{\tau_0} = x \right)$$

Proof of bounds: Let

$$M_t^f = Q_t(X_t, B_t) \quad \text{and} \quad N_t = \int_0^t \star \partial Q f(X_t, B_t) \, dB_t.$$

Then, $|N_0| < |M_0|$ and $[M, M]_t - [N, N]_t$ is non-negative and non-decreasing, so using a Martingale subordination theorem of Bañuelos–Wang,

$$\mathbb{E}[|N_t|^p]^{1/p} \leq (p^* - 1)\mathbb{E}[|M_t|^p]^{1/p}.$$

Isoperimetry Inequality and Poincaré Duality

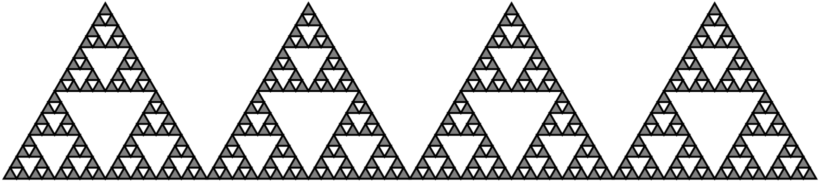
Theorem (Baudoin–K.)

Assuming the Poincaré duality and that $e^{t\vec{\Delta}}\partial = \partial e^{t\Delta}$,

$$\operatorname{Var} f = \sup\left\{\langle f, \partial^* \star g\rangle \;\middle|\; g \in \operatorname{dom}\mathcal{E} \textit{ and } |g| < 1 a.e.\right\}$$

In particular this works for the Sierpinski Gasket.

Further Topics: Fracafolds and Products Fractals



We can build a fractafold by gluing copies of SG together.

Theorem (Baudoin–K.)

The fractafold X admits a Poincaré duality.

The inequality also is preserved by taking products.