Lec 35

9.1.1 Test Procedures for Normal Populations with Known Variances

Assume that both population distributions are normal and the values of σ_1 & σ_2 are known. Thus the difference $\overline{X}_1 - \overline{X}_2$ is also normally distributed, with expected value $\mu_1 - \mu_2$ and standard deviation σ given previously. Standardizing $\overline{X}_1 - \overline{X}_2$ gives the standard normal variable

$$\sigma = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Null Hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0$

Test statistic:

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - \overbrace{(\mu_1 - \mu_2)}^{\Delta_0}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Significance level: α

Alternative Hypothesis:

$H_a : \mu_1 - \mu_2 > \Delta_0$ reject if $Z \geq z_\alpha$

$H_a : \mu_1 - \mu_2 < \Delta_0$ reject if $Z \leq -z_\alpha$

$H_a : \mu_1 - \mu_2 \neq \Delta_0$ reject if $|Z| \geq z_{\alpha/2}$

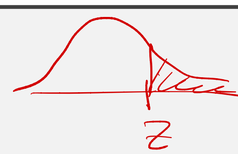
Assumptions:

- Two normal populations
- σ_1 & σ_2 known
- two samples are indep.

P-value for z Test:

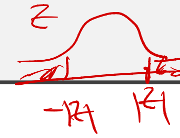
P-value

$H_a : \mu_1 > \mu_2$ $\quad 1 - \Phi(z)$



$H_a : \mu_1 < \mu_2$ $\quad \Phi(z) = 1 - \Phi(-z)$

$H_a : \mu_1 \neq \mu_2$ $\quad 2(1 - \Phi(|z|))$



Example 101. Analysis of a random sample consisting of $m = 20$ specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of $\overline{x} = 29.8$ ksi. A second random sample of $n = 25$ two-sided galvanized steel specimens gave a sample average strength of $\overline{y} = 34.7$ ksi. Assuming that the two yield-strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$, does the data indicate that the corresponding true average yield strengths μ_1 and μ_2 are different? Test at significance level $\alpha = 0.01$.

Solution.

$H_0: \mu_1 = \mu_2 \qquad H_a: \mu_1 \neq \mu_2$

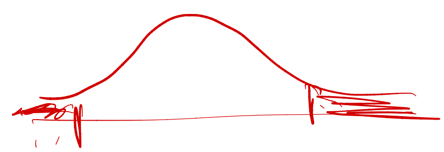
$$Z = \frac{29.8 - 34.7}{\sqrt{\frac{4^2}{20} + \frac{5^2}{25}}} = -3.66$$

$$z_{0.005} = 2.575$$

reject if $|-3.66| \geq 2.575$ ✓

we do reject H_0 in favor of H_a

p-value



$-3.66 \qquad 3.66$

$\leq 2 \times 0.0002$

9.1.2 Large-Sample Tests

The assumptions of normal population distributions and known values of σ_1 and σ_2 are fortunately unnecessary when both sample sizes are large (> 30). In this case, the Central Limit Theorem guarantees that $\overline{X}_1 - \overline{X}_2$ has approximately a ______________ distribution regardless of the underlying population distributions.

Null Hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic:

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Significance level:

Alternative Hypothesis:

same as before

Assumptions:

- samples independent
- $n_1 \geq 30$ $n_2 \geq 30$

9.1.3 Confidence Intervals for $\mu_1 - \mu_2$

When both population distributions are (at least approximately) normal, standardizing $\overline{X}_1 - \overline{X}_2$ gives a random variable Z with a standard normal distribution. Since the area under the z curve between $-z_{\alpha/2}$ and $z_{\alpha/2}$ is $1 - \alpha$, it follows that

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

which implies

$$\bar{X}_1 - \bar{X}_2 \pm z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$ if σ_1 & σ_1 known

$$\bar{X}_1 - \bar{X}_2 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$ if not

This implies that a $100(1-\alpha)\%$ for $\mu_1 - \mu_2$.

Provided that $n_1 > 40$ and $n_2 > 40$, a CI for $\mu_1 - \mu_2$ with a confidence level of $100(1-\alpha)\%$ is

Example 102.

Let μ_1 = the true average tread life for a premium brand of P205/65R15 radial tire, and let μ_2 = the true average tread life for an economy brand of the same size. Independent random samples of tires of each brand were obtained and yielded the summaries below.

	Sample Size	Sample Mean	Sample Std. Dev.
Premium	$n_1 = 45$	$\bar{x}_1 = 42500$	$s_1 = 2200$
Economy	$n_2 = 45$	$\bar{x}_2 = 36800$	$s_2 = 1500$

(a) Do the samples give enough evidence to conclude that μ_1 will exceed μ_2 by more than 5000 miles? Test $H_0: \mu_1 - \mu_2 = 5000$ versus $H_A: \mu_1 - \mu_2 > 5000$ using the level of significance $\alpha = 0.05$.

(b) By how many miles will μ_1 exceed μ_2? Calculate a 90% confidence interval to estimate the true value of $\mu_1 - \mu_2$.

(c) Do the tread lives of the two brands need to be normally distributed for the test of hypotheses and the confidence interval to be valid? Why or why not?

Solution.

$$Z = \frac{(42500 - 36800) - 5000}{\sqrt{\frac{2200^2}{45} + \frac{1500^2}{45}}} = 1.76$$

$z_{0.05} = 1.645$

reject if $1.76 > 1.645$

reject Ho in favor of H_a

$$42500 - 36800 \pm z_{0.05} \cdot \sqrt{\frac{2200^2}{45} + \frac{1500^2}{45}}$$

1.645

c) No b/c

$n_1 \geq 30$ & $n_2 \geq 30$