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Section 8.1: P-value

One way to report the result of a hypothesis test is using p-value.

> **Definition 32.** The P-value is the probability, calculated assuming H_0 is true, of obtaining a value of the test statistic at least as extreme as the value calculated from the available sample data.

A conclusion is reached in a hypothesis testing analysis by comparing the P-value with the specified significant level α.

1. P-value $\leq \alpha \Longrightarrow$ reject H_0
2. P-value $> \alpha \Longrightarrow$ fail to reject

standard normal CDF

> P-value for z Test:
>
> $$\text{P-value} = \begin{cases} 1-\Phi(z) & H_a: \mu > \mu_0 \ (\text{or } \hat{p} > p_0) \\ \Phi(z) & H_a: \mu < \mu_0 \\ 2(1-\Phi(z)) & H_a: \mu \neq \mu_0 \end{cases}$$

Example 92 continued : The melting point of each of 16 samples of a certain brand of hydrogenated vegetable oil was determined, resulting in $\overline{x} = 94.32$. Assume that the distribution of the melting point is normal with $\sigma = 1.2$.

Question: Test $H_0 : \mu = 95$ versus $H_A : \mu \neq 95$ using a two-tailed level 0.01 test.

Solution. Given $n = 16$, $\overline{x} = 94.32$, $\sigma = 1.2$, $X \sim N(\mu,\ \sigma = 1.2)$.So

$$\overline{X} \sim N\left(\mu,\ \frac{\sigma}{\sqrt{n}} = \frac{1.2}{4} = 0.3\right)$$

Step 1: $H_0 : \mu = 95$ versus $H_A : \mu \neq 95$ Note, this is a two tail test.

Step 2: Assume the null is true, i.e.

$$\overline{X} \sim N(95,\ 0.3)$$

Step 3: Since $\alpha = 0.01$, so from the z-table we have the critical values are

$$z_{\alpha/2} = 2.57, \quad -z_{\alpha/2} = -2.57$$

$$Z = \frac{94.32-95}{1.2/\sqrt{16}} = -2.27$$

$$z_{0.005} = 2.57$$

$-2.27 \in [-2.57, 2.57]$

so fail to reject

Thus, the RR = we will reject H_0 is $|z| \geq 2.57$.

Step 4: Test statistic

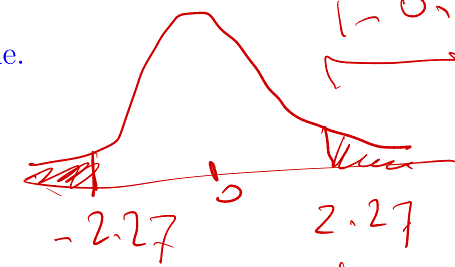
$$z = \frac{\overline{x} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{94.32 - 95}{0.3} = -2.27$$

Step 5: Decision: we fail to reject H_0.

Step 6: Conclusion: Bases on our evidence at $\alpha = 0.01$ or at the 1% significance level, we cannot conclude that the average melting point is different from 95°F.

Let's achieve the same conclusion by calculating the P-value.

$1 - 0.9884$



$Z = -2.27$

P-val. $2 \times (1 - 0.9884)$

$= 0.0232$

b/c > 0.01 fail to reject

$\Phi(2.27) = 0.9884$

Example 93 continued : A manufacturer of sprinkler systems used for fire protection in office buildings claims that the true average system-activation temperature is 130°F. A sample of $n = 9$ systems yields a sample average activation temperature of 131.08°F. If the distribution of activation times is **normal with standard deviation** 1.5°F, does the data contradict the manufacturer's claim at significance level $\alpha = 0.01$?

$\bar{x} = 131.08$

$\mu = 130$

$\sigma = 1.5$

Solution.

Step 1:

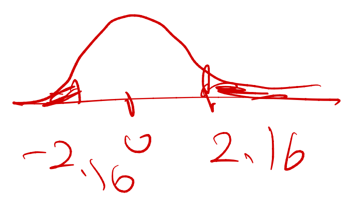
Hypothesis: $H_0 : \mu = 130$ vs $H_A : \mu \neq 130$

Step 2: Assume H_0 is true

Step 3: From the z-table, we have $\Phi(2.575) = 0.995 = 1 - \alpha/2$. Thus, the critical value is $z_{0.005} = 2.575$, the RR for a two-tail test is: reject H_0 if $|Z| > 2.575$.

Step 4: Test statistic

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{131.08 - 130}{1.5/\sqrt{9}} = 2.16$$



Step 5: Decision: since $Z = 2.16 < 2.575$, so we fail to reject H_0.

Step 6: Conclusion: Based on our evidence at α= 0.01, we conclude that the data does not give strong support to the claim that the true average differs from the design value of 130.

P-val. $= 2(1 - \Phi(2.16))$

$= 0.0308 > 0.01$

fail to reject H_0

Let's achieve the same conclusion by calculating the P-value.

Example 94 continued : The desired percentage of SiO_2 in a certain type of aluminous cement is 5.5. To test whether the true average percentage is 5.5 for a particular production facility, 16 independently obtained samples are analyzed. Suppose that the percentage of SiO_2 in a sample is **normally distributed** with $\sigma = 0.3$ and that $\overline{x} = 5.25$.

Does this indicate conclusively that the true average percentage differs from 5.5?

Solution.

Step 1: Hypothesis: $H_0 : \mu = 5.5$ vs $H_A : \mu \neq 5.5$

Step 2: Assume H_0 is true.

Step 3: Since part (a) does not specify a significant level, so we use $\alpha = 0.05$ for this two-tailed test. From the z-table we have $\Phi(1.96) = 0.975$ which give the critical value $z_{0.025} = 1.96$. So RR: we reject H_0 if $|Z| \geq 1.96$.

Step 4: Test statistic

$$Z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{5.25 - 5.5}{0.3/\sqrt{16}} = -3.33$$

Step 5: Decision: since $|Z| = 3.33 > 1.96$, we reject H_0.

Step 6: Conclusion: Based on our evidence at α= 0.05, we conclude that the true average percentage differs from 5.5.

Let's achieve the same conclusion by calculating the P-value.

Example 95 continued : The biological dessert in the Gulf of Mexico called the Dead Zone is a region in which there is very little or no oxygen. Most marine life in the Dead Zone dies or leaves the region. The area of this region varies and is affected by agriculture, fertilizer runoff, and weather. The long-term mean area of the Dead Zone is 5960 square miles. As a result of recent flooding in the Midwest and subsequent runoff from the Mississippi River, researchers believe that the Dead Zone area will increase. A random sample of 50 days was obtained and the sample mean area of the Dead Zone was 6759 mi^2 with a sample standard deviation of 1850 mi^2. Does the sample provide enough evidence to confirm the researchers' belief? Test using $\alpha = 0.025$.

Solution. Given information $\mu = 5960$, $n = 50$, $\overline{x} = 6759$, $s = 1850$, $\alpha = 0.025$.

$\overline{x} = 6759$

$S = 1850$

$n = 50$

$\alpha = 0.025$

Step 1: $H_0 : \mu = 5960$ vs $H_A : \mu > 5960$ (right tail test)

Step 2: Assume H_0 is true.

Step 3: From z-table, we have $\Phi(1.96) = 0.975$, so the critical value $z_{0.025} = 1.96$. We will reject H_0 if Z is greater than 1.96. RR: $Z \geq 1.96$

Step 4:

$\mu_0 = 5960$

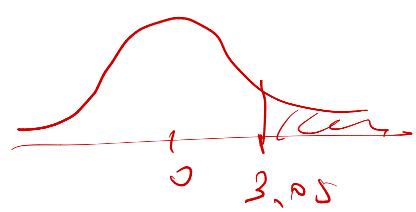
$$z = \frac{\overline{x} - \cancel{\mu_x}}{s/\sqrt{n}} = 3.05$$

Step 5: Decision: we reject H_0.

Step 6: Conclusion: Based on our evidence at $\alpha = 0.025$, we conclude that the area of the Dead zone was increased.

Let's achieve the same conclusion by calculating the P-value.

P-value:



$1 - \Phi(3.05) = 0.0011$

0.9989

< 0.025

Reject H_0

Example 96 continued : Suppose that for a particular application it is required that the true average DCP value for a certain type of pavement be less than 30. The pavement will not be used unless there is conclusive evidence that the specification has been met. A descriptive summary obtained from a sample of $n = 52$ data shows that the sample mean $\overline{x} = 28.76$ and the sample sd $s = 12.2647$. Let's state and test the appropriate hypotheses for the use of the pavement.

Solution.

Step 1: Hypothesis: $H_0 : \mu = 30$ vs $H_A : \mu < 30$

So the pavement will not be used unless the null hypothesis is rejected.

Step 2: Assume the H_0 is true.

Step 3: The significant level is not specified, so we use $\alpha = 0.05$. Since we have a lower-tail test and $\Phi(1.645) = 0.95$, the critical value is $-z_{0.05} = -1.645$. RR: we reject H_0 if $Z \leq -1.645$.

Step 4: Test statistics

$$Z = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{28.76 - 30}{12.2647/\sqrt{52}} = -0.73$$

Step 5: Decision: since $Z = -0.73 > -1.64$, we fail to reject H_0.

Step 6: Conclusion: Based on our evidence at α= 0.05, we conclude that the use of the pavement is not justified.

Let's achieve the same conclusion by calculating the P-value.

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Example 97 continued : Light bulbs of a certain type are advertised as having an average lifetime of 750 hours. The price of these bulbs is very favorable and so a potential customer has decided to go ahead with the purchase unless it can be conclusively demonstrated that the true average lifetime is smaller than what is advertised. A random sample of 20 bulbs was selected and the lifetime of each was recorded. Suppose that the sample mean was 738.4 hours with a sample standard deviation of 41.2 hours. Does the sample provide evidence that the true mean lifetime is less than 750? Assume lifetimes vary according to a normal distribution and test using $\alpha = 0.10$.

Solution. Given information $\mu = 750$, $n = 20$, $\overline{x} = 738.4$, $s = 41.2$, $\alpha = 0.1$.

Step 1: $H_0 : \mu = 750$ vs $H_A : \mu < 750$ (lower-tail test)

Step 2: Assume H_0 is true.

$$T = \frac{738.4 - 750}{41.2/\sqrt{20}} = -1.259$$

Step 3: From t-table, we have $t_{n-1,\alpha} = t_{19,0.1} = 1.328$. Since we have a left tail test, so the critical value is $-t_{19,0.1} = -1.328$. We will reject H_0 if T is smaller than -1.328. RR: $T \leq -1.328$

$t_{19,0.1} = 1.328$

$-1.259 > -1.328$

fail to reject $\quad -t_{\alpha, n-1}$