Lec 32

8.3 The One-Sample t Test

When n is small, the Central Limit Theorem (CLT) can no longer be invoked to justify the use of a large-sample test. Our approach here is ______________________________ ______________________________ and describing test procedures whose validity rests on this assumption.

> Hypothesis test for μ when:
> 1. sample size is small
> 2. X follows normal distribution
> 3. σ is unknown
>
> Then the test statistic
> $$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$
> has a t distribution with $n-1$ degrees of freedom (df).
> Null hypothesis: $H_0 : \mu = \mu_0$
> Significant level: α
> Alternative Hypothesis:
>
> | | | |
> |---|---|---|
> | $H_A : \mu > \mu_0$ | RR: $T \geq t_{n-1,\alpha}$ | upper-tail test |
> | $H_A : \mu < \mu_0$ | RR: $T \leq -t_{n-1,\alpha}$ | lower-tail test |
> | $H_A : \mu \neq \mu_0$ | RR: $\|T\| \geq t_{n-1,\alpha/2}$ | two-tail test |

Example 97. (HW #7) Light bulbs of a certain type are advertised as having an average lifetime of 750 hours. The price of these bulbs is very favorable and so a potential customer has decided to go ahead with the purchase unless it can be conclusively demonstrated that the true average lifetime is smaller than what is advertised. A random sample of 20 bulbs was selected and the lifetime of each was recorded. Suppose that the sample mean was 738.4 hours with a sample standard deviation of 41.2 hours. Does the sample provide evidence that the true mean lifetime is less than 750? Assume lifetimes vary according to a normal distribution and test using $\alpha = 0.10$.

Solution.

$n = 20$ $\quad \bar{X} = 738.4 \quad S = 41.2$

$\alpha = 0.1 \quad H_0 : \mu = 750 \quad H_a : \mu < 750$

$$T = \frac{738.4 - 750}{41.2/\sqrt{20}} = -1.259$$

reject if $T < -t_{19, 0.1}$

$t_{19, 0.1} = 1.328$

$-1.259 \not< -1.328$

fail to reject

Example 98. After a 24-hour smoking abstinence, each of 20 smokers was asked to estimate how much time had elapsed during a 45-second period. The collected elapsed time gives sample mean $\overline{x} = 59.3$ sec and sample sd $s = 9.84$ sec. Assume the data follows a normal distribution. Let's carry out a test of hypotheses at significance level 0.05 to decide whether true average perceived elapsed time differs from the known time 45 sec.

Solution.

$\overline{X} = 59.3 \qquad S = 9.84 \qquad n = 20$

$\alpha = 0.05$

$H_0 : \mu = 45 \qquad H_a : \mu \neq 45$

two-sided H_a

$$T = \frac{59.3 - 45}{9.84/\sqrt{20}} = 6.5$$

$$t_{n-1, \frac{\alpha}{2}} = t_{19, 0.025} = 2.093$$

reject if $|T| > t_{19, 0.025}$

$6.5 > 2.093$ ✓

reject H_0

8.4 Tests Concerning a Population Proportion

Let p denote the proportion of individuals or objects in a population who possess a specified property (e.g., college students who graduate without any debt, or computers that do not need service during the warranty period). If an individual or object with the property is labeled a success (S), then p is proportion of success in pop.

X (the number of S's in the sample) has ~~approximately~~ a Bin (n, p) dist. Furthermore, if n is large [$np \geq 10$ and $n(1-p) \geq 10$], both X and the estimator $\hat{p} = \frac{X}{n}$ are approximately normal.

proportion in sample

For the estimator $\hat{p} = \frac{X}{n}$, we have

mean $= p$ variance $= \frac{p(1-p)}{n}$

$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$

It follows that when n is large and $H_0 : p = p_0$ is true, the test statistic

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

has approximately a standard normal dist.

Null hypothesis: $H_0 : p = p_0$
Test statistic

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Significance level : α
Alternative Hypothesis:

$H_A : p > p_0$	RR: $Z \geq z_\alpha$	upper-tailed test
$H_A : p < p_0$	RR: $Z \leq -z_\alpha$	lower-tailed test
$H_A : p \neq p_0$	RR: $\lvert Z \rvert \geq z_{\alpha/2}$	two-tailed test

These test procedures are valid provided that

$$np_0 \geq 10 \quad \text{and} \quad n(1-p_0) \geq 10$$

Example 99. The use of a phone to text during an exam is a serious breach of conduct. One article reported that 27 of the 267 students in a sample admitted to doing this. Can it be concluded at significance level 0.001 that more than 5% of all students in the population sampled had texted during an exam?

Solution. H_0 $p = 0.05$ $H_a : p > 0.05$ $n = 267$

$\hat{p} = \frac{27}{267}$ $\alpha = 0.001$

$$Z = \frac{\frac{27}{267} - 0.05}{\sqrt{\frac{0.05(1-0.05)}{267}}} = 3.84$$

$$267 \times 0.05 \geq \frac{200}{20} = 10$$

$$267 \times (1 - 0.05) \geq \text{"} \geq 10$$

$z_{0.001} = 3.1$ reject if $Z > z_{0.001}$

$3.84 > 3.1$

reject H_0 in favor of H_a

Example 100. A plan for an executive travelers' club has been developed by an airline on the premise that 5% of its current customers would qualify for membership. A random sample of 500 customers yielded 40 who would qualify. Using this data, test at level 0.01 the null hypothesis that the company's premise is correct against the alternative that it is not correct.

Solution.