Lec 28

7.3 Intervals Based on a Normal Population Distribution

We know how to estimate μ when n is large. But what should we do when n is small?

Assumptions:

(1) The population of interest has to be normal. (or close to normal)

(2) Population mean, μ, is unknown.

(3) Population SD, σ, is unknown.

Theorem 7.1. When $\overline{X}$ is the mean of a random sample of size n from a __________ distribution with mean μ, then the random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

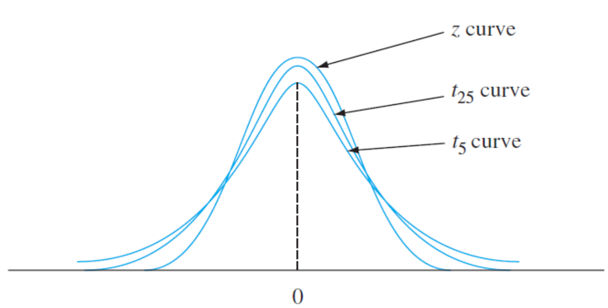
has a probability distribution called t - distribution with n-1 degrees of freedom

Properties of t distribution:

Let t_ν denote the t distribution with ν df.

1. Each t_ν curve is bell-curve shaped centered at 0.
2. Each t_ν curve is more spread out than the standard normal.
3. As ν $\nearrow$, the spread of the corresponding t_ν curve $\searrow$.
4. As $\nu \to \infty$, the sequence of t_ν curves approaches the standard normal curve.
Usually when $n > 30$, then we can use the z curve.

ν



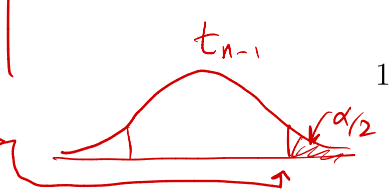
t_ν and z curves

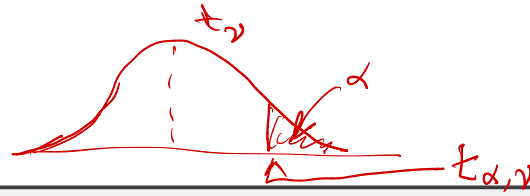
Remember:

$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ $\sigma \to S$

$\updownarrow$

$t_{\frac{\alpha}{2}, n-1}$





Notation:
Let $t_{\alpha,\nu}$ = the number on the $x-$axix for which the area under the t_ν curve to the right of $t_{\alpha,\nu}$ is α. $t_{\alpha,\nu}$ is called a Critical value

Example 85. (HW question 8) Given the sample size $n = 15$ and the confidence level is 90%. Assume σ is unknown. Find the $t_{\alpha/2}$ critical value for the confidence interval $\boxed{\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}}$.

Solution. Given : $\bar{X}$, S, $n = 15$, $\alpha = 0.1$

$t_{\frac{\alpha}{2},n-1} = t_{0.05,14} = 1.761$

$\bar{X} \pm 1.761 \frac{S}{\sqrt{15}}$

Let $\overline{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal population with mean μ. Then a $100(1-\alpha)\%$ confidence interval for μ is

$$\bar{X} \pm t_{\frac{\alpha}{2},n-1} \frac{S}{\sqrt{n}}$$

If instead we want a CUB: $\bar{X} + t_{\alpha,n-1} \frac{S}{\sqrt{n}}$ or CLB: $\bar{X} - t_{\alpha,n-1} \frac{S}{\sqrt{n}}$

Example 86. (HW question 9) A random sample of 10 brands of vanilla yogurt was selected and the calorie count per serving was recorded, resulting in the following data:

$$130, 160, 150, 120, 120, 110, 170, 160, 110, 90$$

Calculate a 90% confidence interval to estimate the true mean calorie count.

Solution.

$\bar{X} = 132$

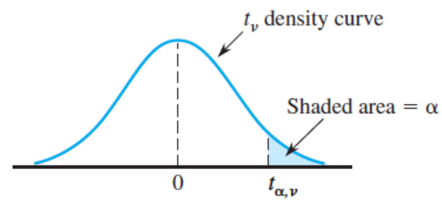
$S = 26.583$

$\alpha = 0.1$

$t_{0.05,9} = 1.833$

$132 \pm 1.833 \cdot \frac{26.583}{\sqrt{10}}$

Table A.5 Critical Values for t Distributions



	α						
ν	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015

Summary:

n large ($n \geq 30$):
- σ known: $\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
- σ unknown: $\bar{X} \pm z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$

  $\left(\hat{p} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\right)$

n small & pop. is normal or close to normal:
- σ known: $\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
- σ unknown: $\bar{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$

n small & pop. is not normal: don't know!