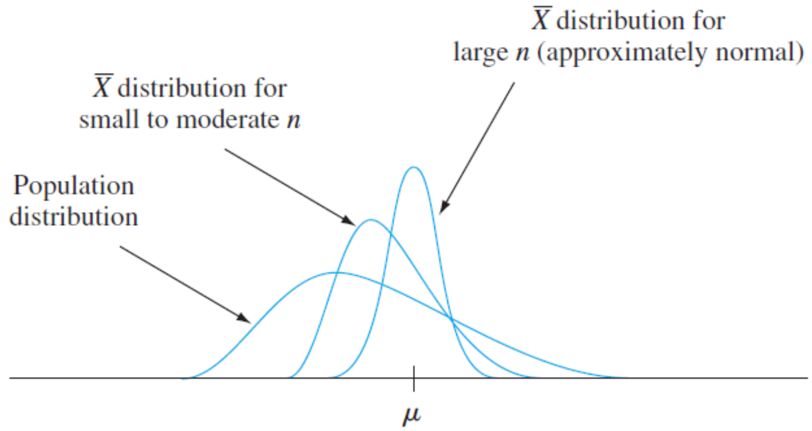
Lec 24



The Central Limit Theorem illustrated

Figure above illustrates the Central Limit Theorem. According to the CLT, when n is large and we wish to calculate a probability such as $P(a \leq \overline{X} \leq b)$, we need only "pretend" that $\overline{X}$ is normal, standardize it, and use the normal table.

Example 76. Suppose that a certain type of cable strength is normally distributed with mean $\mu = 450$lb and sd$=\sigma = 50$lb. $X \sim N(450, 50^2)$

(a) Find the probability that the strength of the cable is greater than 536lbs.

$$P(X \geq 536) = P\left(Z \geq \frac{536-450}{50}\right)$$

$$= 1 - 0.9573 = 0.0427$$

(b) Let $\overline{X}$ = the mean strength for a sample of 9 cables. Find the shape, mean and sd of $\overline{X}$.

pdf

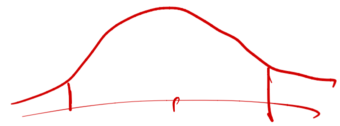
$\frac{\sigma^2}{n}$

$$N\left(450, \frac{50^2}{3^2}\right)$$

$\frac{\sigma}{\sqrt{n}} \rightarrow$ stdev $= \frac{50}{\sqrt{9}} = \frac{50}{3}$

(c) Find the probability that the sample mean of 9 cables will be between 423lbs and 480 lbs.

$$P(423 \le \overline{X} \le 480)$$



$$= P\left(\frac{423-450}{50/3} \le Z \le \frac{480-450}{50/3}\right)$$

$$= 0.9641 - (1 - 0.9474) = 0.9115$$

(d) Suppose we have 35 cables, find $P(\overline{X} < 428)$.

$$P(\overline{X} < 428) = P\left(Z < \frac{428-450}{50/\sqrt{35}}\right)$$

$$= 0.0197$$

(e) Suppose that the population is not normal (or unknown)

- Can we still solve part(a)? ✗
- Can we still solve part (c)? ✗
- Can we still solve part (d)? ✓

5.5 The Distribution of a Linear Combination

Definition 31. Given a collection of n random variables $X_1, \cdots, X_n$ and n numerical constants $a_1, \cdots, a_n$, the random variable

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is called a **linear combination** of the X_i's.

Taking $a_1 = a_2 = \cdots = a_n = 1$ gives $X_1 + \cdots + X_n$

Taking $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$ gives $\dfrac{X_1 + \cdots + X_n}{n}$

Let $X_1, X_2, \cdots, X_n$ have mean values μ_1, μ_2, $\cdots$, μ_n, respectively and variances σ_1^2, σ_2^2, $\cdots$, σ_n^2, respectively.

1. Whether or not the X_i's are independent,

$$E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]$$

2. If $X_1, \cdots, X_n$ are independent,

$$Var(a_1 X_1 + \cdots + a_n X_n) = a_1^2 Var(X_1) + \cdots + a_n^2 Var(X_n)$$

The Case of Normal Random Variables

If $X_1, X_2, \cdots, X_n$ are independent normally distributed random variables (with possibly different means and variances), then any linear combination of X_i's is normally distributed with mean and variance as given earlier.

Example 77. (From homework)

I have two errands to take care of on campus. Let X_1 and X_2 represent the times that it takes for the first and second errands, respectively. Let X_3 = the total time in minutes that I spend walking to and from my office and between the errands. Suppose that X_1, X_2, X_3 are independent and normally distributed with μ_1= 15, $\sigma_1 = 4$, $\mu_2 = 5$, $\sigma_2 = 1$, μ_3= 12, and σ_3=3.

$X_1 \sim N(15, 16)$ $X_2 \sim N(5, 1)$ X_1, X_2 indep.

$X_3 \sim N(12, 9)$

(a) Find the chance that the total time I am away from my office is less than 45 minutes, i.e. find $P(X_1 + X_2 + X_3 < 45)$.

$$X_1 + X_2 + X_3 \sim N(\overset{15+5+12}{32}, \underset{16+1+9=26}{26})$$

$$P(\underbrace{X_1 + X_2 + X_3 < 45}) = P\left(Z < \frac{45-32}{\sqrt{26}}\right)$$

$\simeq 0.9946$

(b) Find the probability of the average amount of time it takes less than 12 minutes.

$$P\left(\underbrace{\frac{X_1+X_2+X_3}{3}} \le 12\right) = \cdots = 0.7823$$

$$N\left(\frac{15+5+12}{3}, \frac{16+1+9}{9}\right)$$

$3^2 \rightarrow 9$

(c) Find the probability $P(X_1 - X_3 > 0)$.

(or $P(X_1 > X_3)$)

$$N(15-12, \underbrace{1^2 \cdot 16 + (-1)^2 \cdot 9}_{16+9})$$

$$N(3, 25)$$

$$P(X_1 - X_3 > 0) = P\left(Z > \frac{0-3}{5}\right) \doteq 0.7257$$