Lec 14

$n=1$ $\quad Bin(1,p) = Ber(p)$ $\quad E[X^2] = p\times 1 + (1-p)\times 0 = p$

$E[Ber(p)] = 1\times p + 0\times(1-p) = p$

The Mean and Variance of X $\quad var(Ber(p)) = p - p^2 = p(1-p)$

> If $X \sim Bin(n,p)$, then
> $$E(X) = np$$
> $$\text{Var}(X) = np(1-p)$$
> $$\sigma_X = \sqrt{np(1-p)}$$
> where $q = 1 - p$.

Example 43. If 75% of all purchases at a certain store are made with a credit card and X is the number among ten randomly selected purchases made with a credit card, then

$X \sim Bin(10, 0.75)$

0	1
$1-p$	p

$E[X] = 10 \times 0.75 = 7.5$

$Var(X) = 10 \times 0.75 \times 0.25 = 1.875$

$\sigma_X = \sqrt{1.875} = 1.37$

p(within 1 standard dev. from mean)

$P(7.5 - 1.37 \le X \le 7.5 + 1.37)$

$= P(6.13 \le X \le 8.87)$

$= P(X = 7 \text{ or } 8) = \binom{10}{7} \times (0.75)^7 (0.25)^3 + \binom{10}{8} \times (0.75)^8 (0.25)^2$

$= 0.532$

Example 44. (Exercise 46 on Page 123) Compute the following binomial probabilities directly from the formula for $b(x; n, p)$:

a. $b(3; 8, 0.35)$

b. $b(5; 8, 0.6)$

c. $P(3 \le X \le 5)$ when $n = 7$ and $p = 0.6$

d. $P(1 \le X)$ when $n = 9$ and $p = 0.1$

Solution.

Bin(8, 0.35)

$P(X=3) = \binom{8}{3} \times (0.35)^3 \times 0.65^5 = 0.2787$

Bin(8, 0.6)

$P(X=5) = \binom{8}{5} 0.6^5 \, 0.4^3$

$\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8 \times 7 \times 6}{\cancel{3 \times 2 \times 1}} = 8 \times 7 = 56$

$\binom{8}{5} = \binom{8}{3} = 56$

$P(X=3) + P(X=4) + P(X=5)$

if using table:

$P(X \le 5) - P(X \le 2)$

$1 - P(X=0)$

$= 1 - 1 \times (0.1)^0 (0.9)^9 = 0.6126$

$\binom{9}{0}$

Example 45. (Exercise 47 on Page 123) The article "Should You Report That Fender - Bender? " reported that 7 in 10 auto accidents involve a single vehicle. Suppose 15 accidents are randomly selected. Use Appendix Table A.1 to answer each of the following questions.

a. What is the probability that at most 4 involve a single vehicle?

b. What is the probability that exactly 4 involve a single vehicle?

c. What is the probability that exactly 6 involve multiple vehicles?

d. What is the probability that between 2 and 4, inclusive, involve a single vehicle?

e. What is the probability that at least 2 involve a single vehicle?

f. Find the mean and standard deviation of X.

Solution.

Do this

↓ takes values 0, 1, 2, 3, ... —
$x \rightarrow$

3.6 The Poisson Probability Distribution

Definition 15. A discrete random variable X is said to have a Poisson distribution with parameter $\mu > 0$ if the pmf of X is

$$p(x;\mu) = e^{-\mu}\frac{\mu^x}{x!}$$

NOTE: The Poisson distribution spreads probability over all non-negative integers (in contrast to the binomial distribution), an infinite number of possibilities.

Appendix Table A.2 contains the Poisson cdf $F(x;\mu)$ for $\mu = 0.1, 0.2, \cdots, 1, 2, \cdots, 10$, 15, and 20. Alternatively, many software packages will provide $F(x;\mu)$ and $p(x;\mu)$ upon request.

Example 46. Let X denote the number of traps in a particular type of metal transistor, and suppose it has a Poisson distribution with $\mu = 2$.

The probability that there are exactly three traps is

$$P(X=3) = e^{-2}\frac{2^3}{3!}$$

and the probability that there are at most three traps is

$$P(X \le 3) = 0.857$$

$$e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} + e^{-2}\frac{2^3}{3!}$$

1

This latter cumulative probability is found at the intersection of the ________ and the ________ of Appendix Table A.2, whereas $p(3;2) = F(3;2) - F(2;2) = 0.857 - 0.677 = 0.180$, the difference between two consecutive entries in the $\mu = 2$ column of the cumulative Poisson table.

$X \sim Poi(7)$ $\quad P(X \le 5) = 0.301$

Table A.2 Cumulative Poisson Probabilities *(cont.)*

$$F(x;\mu) = \sum_{y=0}^{x} \frac{e^{-\mu}\mu^{y}}{y!}$$

						μ					
x	**2.0**	**3.0**	**4.0**	**5.0**	**6.0**	**7.0**	**8.0**	**9.0**	**10.0**	**15.0**	**20.0**
0	.135	.050	.018	.007	.002	.001	.000	.000	.000	.000	.000
1	.406	.199	.092	.040	.017	.007	.003	.001	.000	.000	.000
2	.677	.423	.238	.125	.062	.030	.014	.006	.003	.000	.000
3	.857	.647	.433	.265	.151	.082	.042	.021	.010	.000	.000
4	.947	.815	.629	.440	.285	.173	.100	.055	.029	.001	.000
5	.983	.916	.785	.616	.446	.301	.191	.116	.067	.003	.000
6	.995	.966	.889	.762	.606	.450	.313	.207	.130	.008	.000
7	.999	.988	.949	.867	.744	.599	.453	.324	.220	.018	.001
8	1.000	.996	.979	.932	.847	.729	.593	.456	.333	.037	.002
9		.999	.992	.968	.916	.830	.717	.587	.458	.070	.005
10		1.000	.997	.986	.957	.901	.816	.706	.583	.118	.011
11			.999	.995	.980	.947	.888	.803	.697	.185	.021
12			1.000	.998	.991	.973	.936	.876	.792	.268	.039
13				.999	.996	.987	.966	.926	.864	.363	.066
14				1.000	.999	.994	.983	.959	.917	.466	.105
15					.999	.998	.992	.978	.951	.568	.157

If X has a Poisson distribution with parameter μ, then

$$E(X) = \mu$$
$$\mathrm{Var}(X) = \mu$$

$\sigma_X = \sqrt{\mu}$

Example 47. (Example 46 continued)

$E[X] = 2$

$\mathrm{Var}(X) = 2 \quad \sigma_X = \sqrt{2}$

$P(2-\sqrt{2} \le X \le 2+\sqrt{2})$

$= P(0.6 \le X \le 3.4)$

$= P(X = 1, 2, 3)$

Example 48. Suppose the number of accidents per month at an industrial plant has a Poisson distribution with mean 2.6. If we denote Y = the number of accidents per month,

then ____________________

(a) Find the probability that there will be 4 accidents in the next month.

$P(X \le 3)$

$- P(X = 0)$

$= 0.857$

e^{-2}

$- 0.135$