The heat equation with “vertical” oscillatory Neumann data

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Vertical oscillatory Neumann data

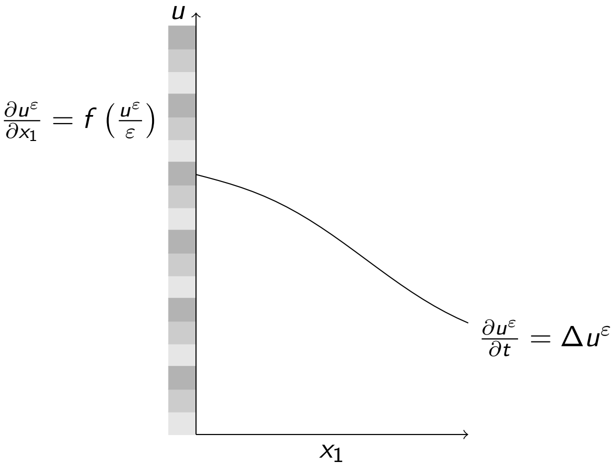
Consider the boundary data homogenization problem

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon & \text{in } B_1^+ \\ \frac{\partial u^\varepsilon}{\partial x_1} = f(\frac{u^\varepsilon}{\varepsilon}) & \text{on } B_1'. \end{cases} \tag{P_ε}$$

Here $B_1^+ := B_1 \cap \{x_1 > 0\}$ is the right half-ball in $\mathbb{R}^d$ and $B_1' = B_1 \cap \{x_1 = 0\}$. The medium f is 1-periodic.

This is a simple model for an evolution of an interface in a heterogeneous medium, especially we are thinking of contact lines where the physical dimension is $d + 1 = 3$.

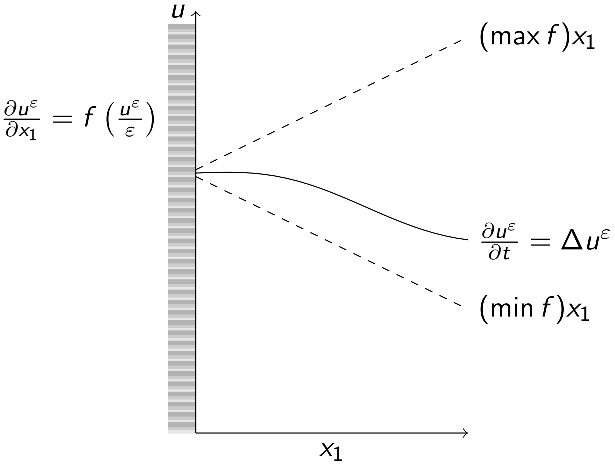
Diagram



u
$\frac{\partial u^\varepsilon}{\partial x_1} = f\left(\frac{u^\varepsilon}{\varepsilon}\right)$
$\frac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon$
x_1

Pinning phenomenon in $d = 1$

Normal slopes in the interval $[\min f, \max f]$ are *pinned*.



Pinning phenomenon in $d = 1$

So the limit $u_\varepsilon \to u$ as $\varepsilon \to 0$ will solve

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in} \quad B_1^+$$

and

$$\frac{\partial u}{\partial x_1} \in [\min f, \max f] \quad \text{on} \quad B_1'.$$

But there is a bit more to it...

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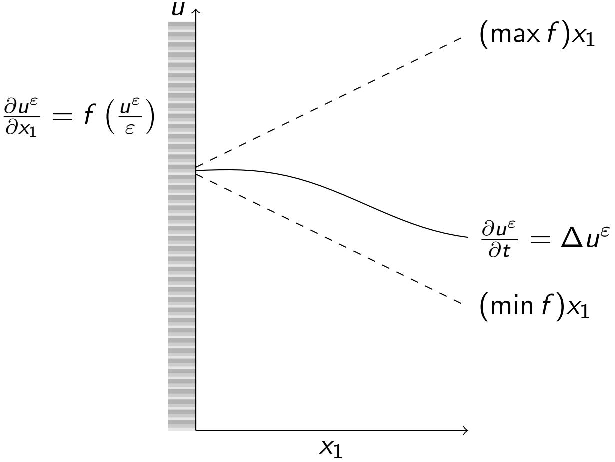
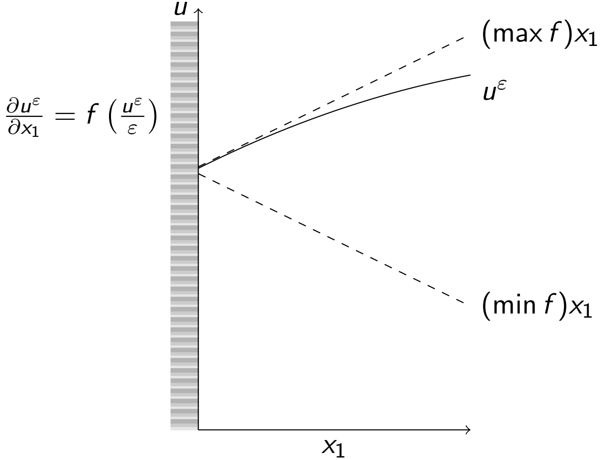


Pinning phenomenon in $d = 1$

Advancing only at the maximal slope



Pinning phenomenon in $d = 1$

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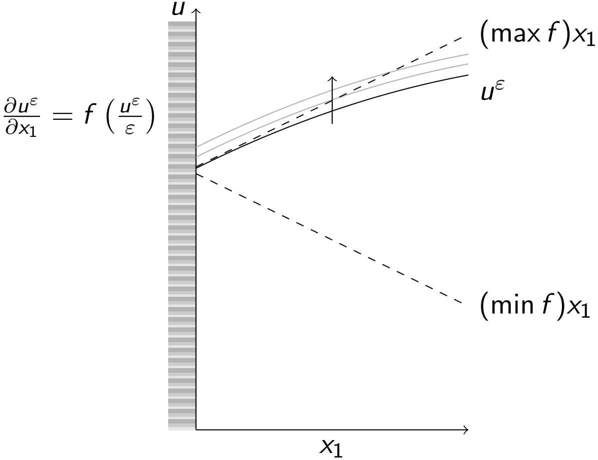
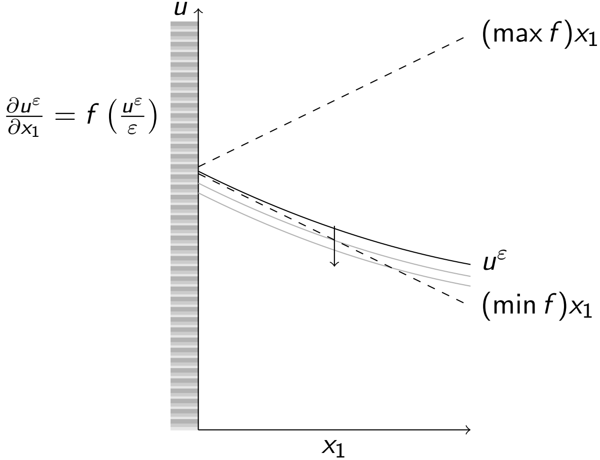


Pinning phenomenon in $d = 1$

Receding only at the minimal slope



Rate-independent Neumann condition in 1-d

In $d = 1$ the associated homogenized problem is the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad B_1^+$$

with the boundary condition on B_1'

- (pinned slope) $\min f \leq \frac{\partial u}{\partial x_1} \leq \max f$
- (advancing slope condition) if $\frac{\partial u}{\partial t} > 0$ then $\frac{\partial u}{\partial x_1} \geq \max f$
- (receding slope condition) if $\frac{\partial u}{\partial t} < 0$ then $\frac{\partial u}{\partial x_1} \leq \min f$

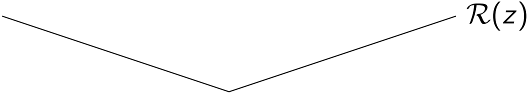
Rate-independent Neumann condition 1-d

This problem can be written compactly, and in a way which respects the energetic structure:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } B_1^+ \times (0,\infty) \\ \frac{\partial u}{\partial x_1} \in \partial \mathcal{R}(\frac{\partial u}{\partial t}) & \text{on } B_1' \times (0,\infty), \end{cases} \tag{1}$$

where $\partial \mathcal{R}$ is the subdifferential of the one-homogeneous *dissipation rate functional*

$$\mathcal{R}(z) = (\min f) \min\{z, 0\} + (\max f) \max\{z, 0\}.$$



Energetic interpretation

Such flows arise in the theory of rate independent systems and modelling of dry friction. For example the minimizing movements scheme

$$v_{(k+1)\tau} = \text{argmin}\Big\{ \int_{B_1^+} |\nabla v|^2 \, dx + \frac{1}{2\tau}\|v - v_{k\tau}\|^2_{L^2(B_1^+)} \\ \cdots + \int_{B_1'} \mathcal{R}(v - v_{k\tau}) \, dx' \Big\}$$

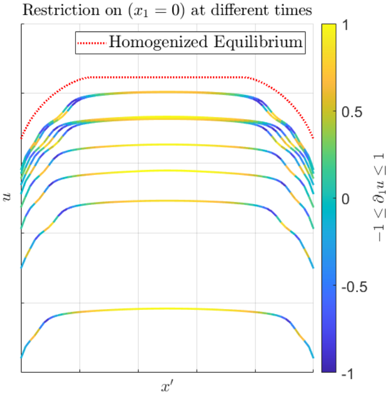
limits, as $\tau \to 0$, to the flow

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v & \text{in } B_1^+ \times (0,\infty) \\ \frac{\partial v}{\partial x_1} \in \partial\mathcal{R}(\frac{\partial v}{\partial t}) & \text{on } B_1' \times (0,\infty). \end{cases} \tag{2}$$

These energetic evolutions are (partially) motivated by the homogenization description above, but only derived by a rigorous limit of this type in $d = 1$.

Anisotropic pinning in higher dimensions

In dimensions $d \geq 2$ the homogenization limit for this problem is in fact much more singular.



Anisotropic pinning in higher dimensions

In dimensions $d \geq 2$ the homogenization limit for this problem is in fact much more singular

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } B_1^+ \times (0,\infty) \\ \frac{\partial u}{\partial x_1} \in \partial \mathcal{R}(\frac{\partial u}{\partial t}; \nabla' u) & \text{on } B_1' \times (0,\infty). \end{cases} \qquad (P_{hom})$$

The rate function now depends on the tangential gradient

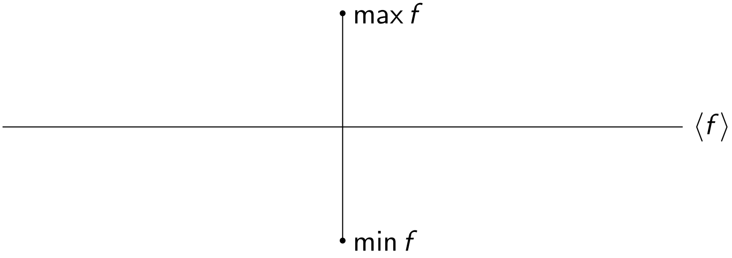
$$\mathcal{R}(z;p') = \begin{cases} \langle f \rangle & p' \neq 0 \\ (\min f) \min\{z,0\} + (\max f) \max\{z,0\} & p' = 0 \end{cases}$$

Pinning interval

The rate function now depends on the tangential gradient

$$\mathcal{R}(z;p') = \begin{cases} \langle f \rangle & p' \neq 0 \\ (\min f)\min\{z,0\} + (\max f)\max\{z,0\} & p' = 0 \end{cases}$$

Plot of $\partial\mathcal{R}(0,p')$, the *pinning interval*, as a function of p':



Intuitive description

As before the interior PDE is the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \ \text{ in } \ B_1^+$$

with the boundary condition on B_1'

- (pinned slope: non-laminar) If $\nabla' u \neq 0$ then $\frac{\partial u}{\partial x_1} = \langle f \rangle$.
- (pinned slope: laminar) If $\nabla' u = 0$ then $\min f \leq \frac{\partial u}{\partial x_1} \leq \max f$
- (advancing slope condition) if $\frac{\partial u}{\partial t} > 0$ then $\frac{\partial u}{\partial x_1} \geq \langle f \rangle$ and, "somewhere on the flat part", $\frac{\partial u}{\partial x_1} \geq \max f$
- (receding slope condition) similar to advancing condition

Main results

Homogenized problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } B_1^+ \times (0,\infty) \\ \frac{\partial u}{\partial x_1} \in \partial \mathcal{R}(\frac{\partial u}{\partial t}; \nabla' u) & \text{on } B_1' \times (0,\infty). \end{cases} \qquad (P_{hom})$$

Theorem (F. and Huang, 2025, arXiv)

Comparison principle holds for the problem (P_{hom}).

Theorem (F. and Huang, 2025, arXiv)

Solutions u^ε *of* (P_ε) *with continuous boundary data* $g(x,t)$ *on* $\partial B_1 \cap \mathbb{R}^d_+$ *converge locally uniformly on* $\overline{B_1^+} \times [0,\infty)$ *to the unique solution* u *of* (P_{hom}).

Brief comment on literature

Some literature (incomplete sorry!):

- Closest work, by Caffarelli, Lee, and Mellet (CPAM, 2006) studies heat propagation free boundary problem

$$\partial_t u - \Delta u = 0 \text{ in } \{u > 0\}, \ |\nabla u| = f(x/\varepsilon) \text{ on } \partial\{u > 0\}$$

  and related reaction diffusion singular limits. Homogenization only proved in $d = 1$ due to lack of sufficient barriers / comparison in higher dimensions. Our work partially resolves this issue, partially since the medium is still laminar.
- Other interface homogenization problems (i.e. curvature flows, or Allen-Cahn-type) have also been addressed in higher dimensions by taking advantage of laminar setting, but they are not as closely related. Barles, Cesaroni, Novaga (SIMA, 2011), Gao and Kim (ARMA, 2019), Morfe (ARMA, 2022).

Outline of the proof of homogenization

Proof of homogenization follows a well-known paradigm in nonlinear PDE theory using perturbed test functions (Evans, '92) and the method of half-relaxed limits (Barles and Perthame, '87).

- Construction of plane-like solutions and pinning interval (works for general $f(x, u)$ periodic on $\mathbb{R}^{d+1}$)
- Show that $\limsup^* u^\varepsilon$ and $\liminf_* u^\varepsilon$ are respectively USC subsolution and LSC supersolution of limit problem
- Apply comparison principle for limit problem conclude local uniform convergence of u^ε

Definition of viscosity solutions

An USC $u(x, t)$ is a subsolution of (P_{hom}) if:

- (local stability) In the viscosity sense

$$\frac{\partial u}{\partial x_1}(x_0, t_0) \geq \begin{cases} \min f & \nabla' u(x_0, t_0) = 0 \\ \langle f \rangle & \text{else.} \end{cases}$$

- (transversal advancing condition) In the viscosity sense

$$\text{if } \frac{\partial u}{\partial t}(x_0, t_0) > 0 \text{ then } \frac{\partial u}{\partial x_1}(x_0, t_0) \geq \langle f \rangle.$$

- (laminar advancing condition) If $\phi(x, t)$ crosses u from above in some region U with strict ordering on the parabolic boundary and if $\nabla' \phi \equiv 0$ in U, then

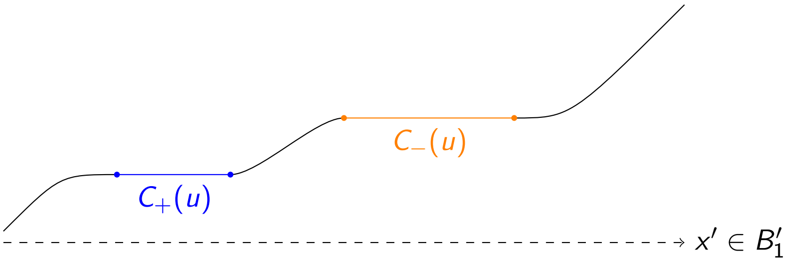
$$\frac{\partial \phi}{\partial x_1}(x_0, t_0) \geq \max f.$$

Contact sets

Define the *contact sets*

$$\mathcal{C}_{\pm}(u) := \{(x,t) \in B_1' \times (0,\infty) : \pm \frac{\partial u}{\partial x_1} > \langle f \rangle\}$$

these are open sets where u has zero tangential gradient.
Connected to thin obstacle problem.



Open questions

Probably do-able

- Similar comparison principle for heat propagation free boundary problems.

The following are quite difficult, but interesting, open issues in my opinion.

- Formal but no rigorous energetic interpretation of the limit evolution, i.e. limit of minimizing movements scheme.
- General $f(x, u)$ periodic on $\mathbb{R}^{d+1}$. Issues both in the construction of enough correctors, and in comparison for the limit PDE.
- Random media.

Thanks for your attention!

Sketch of proof of comparison

Crossing location

Subsolution u crosses supersolution v from below for first time at (x_0, t_0) on the Neumann boundary B_1'. First derivative test plus a perturbation argument to arrive at

$$\frac{\partial u}{\partial x_1}(x_0, t_0) < \frac{\partial v}{\partial x_1}(x_0, t_0) \text{ and } \frac{\partial u}{\partial t}(x_0, t_0) > \frac{\partial v}{\partial t}(x_0, t_0)$$

- (Case 1) $(x_0, t_0) \notin \mathcal{C}_-(u) \cup \mathcal{C}_+(v)$
- (Case 2) $(x_0, t_0) \in \mathcal{C}_-(u) \cap \mathcal{C}_+(v)$
- (Case 3) $(x_0, t_0) \in \mathcal{C}_-(u) \Delta \mathcal{C}_+(v)$

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$$\frac{\partial u}{\partial x_1}(x_0, t_0) < \frac{\partial v}{\partial x_1}(x_0, t_0) \text{ and } \frac{\partial u}{\partial t}(x_0, t_0) > \frac{\partial v}{\partial t}(x_0, t_0)$$

- (Case 1) $(x_0, t_0) \notin \mathcal{C}_-(u) \cup \mathcal{C}_+(v)$
  - Then, by definition of contact sets,

$$\frac{\partial u}{\partial x_1}(x_0, t_0) \geq \langle f \rangle \geq \frac{\partial v}{\partial x_1}(x_0, t_0),$$

    contradicting first derivative test.
- (Case 2) $(x_0, t_0) \in \mathcal{C}_-(u) \cap \mathcal{C}_+(v)$
- (Case 3) $(x_0, t_0) \in \mathcal{C}_-(u) \Delta \mathcal{C}_+(v)$

(Case 2) $(x_0, t_0) \in \mathcal{C}_-(u) \cap \mathcal{C}_+(v)$

From perturbation argument and derivative tests either $\frac{\partial u}{\partial t}(x_0, t_0) > 0$ or $\frac{\partial v}{\partial t}(x_0, t_0) < 0$. Say $\frac{\partial u}{\partial t}(x_0, t_0) > 0$, then apply:

- (transversal advancing condition) In the viscosity sense

$$\text{if } \frac{\partial u}{\partial t}(x_0, t_0) > 0 \text{ then } \frac{\partial u}{\partial x_1}(x_0, t_0) \geq \langle f \rangle.$$

But this contradicts $(x_0, t_0) \in \mathcal{C}_-(x_0, t_0)$. Similarly if $\frac{\partial v}{\partial t}(x_0, t_0) < 0$ apply the transversal receding condition of v.

(Case 3) $(x_0, t_0) \in \mathcal{C}_-(u) \Delta \mathcal{C}_+(v)$

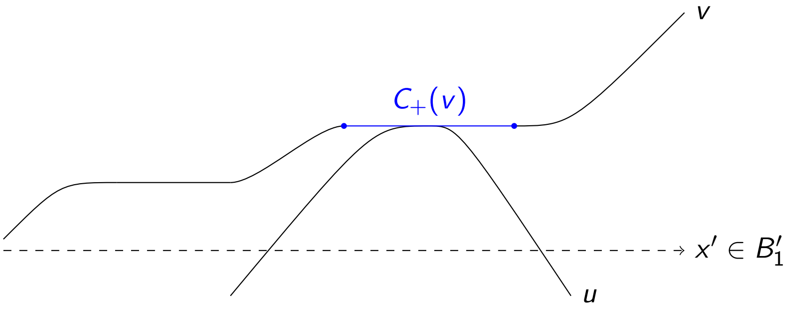
Let's say that $(x_0, t_0) \in \mathcal{C}_+(v) \setminus \mathcal{C}_-(u)$. We can rule out or reduce to a previous case. We will use

- (laminar advancing condition) If $\phi(x, t)$ crosses u from above in some parabolic cylinder $U \times (s, t_0]$ with $U \subset\subset B_1$ and strict ordering on the parabolic boundary and if $\nabla' \phi \equiv 0$, then

$$\frac{\partial \phi}{\partial x_1}(x_0, t_0) \geq \max f.$$

(Case 3) $(x_0, t_0) \in \mathcal{C}_-(u)\Delta\mathcal{C}_+(v)$

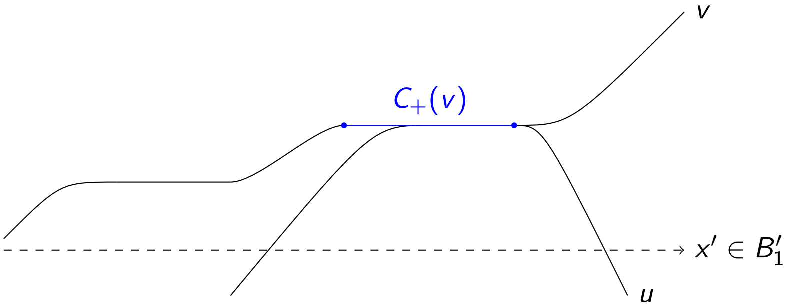
Let's say that $(x_0, t_0) \in \mathcal{C}_+(v) \setminus \mathcal{C}_-(u)$. We can rule out or reduce to a previous case. Proof by picture:



If $\partial_t v > 0$ then can apply the laminar advancing condition, using v as a "one-dimensional" test function. If $\partial_t v < 0$ contradiction similar to case 2.

(Case 3) $(x_0, t_0) \in \mathcal{C}_-(u) \Delta \mathcal{C}_+(v)$

Let's say that $(x_0, t_0) \in \mathcal{C}_+(v) \setminus \mathcal{C}_-(u)$. We can rule out or reduce to a previous case. Proof by picture:



If there is no strict ordering on the edge of the facet can reduce to either Case 1 or Case 2.