One and two phase minimizing free boundaries in periodic media

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Partly based on joint work with Farhan Abedin (Lafayette College)

One- and two-phase problems

Alt-Caffarelli energy functional

$$\mathcal{J}_0(u; U) = \int_U |\nabla u|^2 + q_+^2 \mathbf{1}_{\{u>0\}} + q_-^2 \mathbf{1}_{\{u\leq 0\}} \, dx.$$

Assume $q_+^2 - q_-^2 = 1 > 0$, the case $q_- = 0$ is the one-phase functional.

Local minimizers solve the Bernoulli free boundary problem

$$\begin{cases} -\Delta u = 0 & \text{in } \{|u| > 0\} \cap U \\ |\nabla u_+|^2 - |\nabla u_-|^2 = q_+^2 - q_-^2 & \text{on } \partial\{u > 0\} \cap U. \end{cases}$$

Regularity theory of the free boundary has been a popular topic since foundational works of Alt, Caffarelli and Friedman.

Periodic medium

Alt-Caffarelli energy functional in heterogeneous medium

$$\mathcal{J}_\varepsilon(u;U) = \int_U a(\tfrac{x}{\varepsilon})\nabla u \cdot \nabla u + Q_+(\tfrac{x}{\varepsilon})^2 \mathbf{1}_{\{u>0\}} + Q_-(\tfrac{x}{\varepsilon})^2 \mathbf{1}_{\{u\leq 0\}} \, dx.$$

Media $a(y)$ symmetric and $Q(y)$ are elliptic and $\mathbb{Z}^d$-periodic. It will be convenient to normalize and assume that the homogenized matrix satisfies $\bar{a} = \mathrm{id}$.

Local minimizers solve a Bernoulli free boundary problem

$$\begin{cases} -\nabla \cdot (a(\tfrac{x}{\varepsilon})\nabla u) = 0 & \text{in } \{|u| > 0\} \cap U \\ |\nabla_a u_+|^2 - |\nabla_a u_-|^2 = Q_+^2(\tfrac{x}{\varepsilon}) - Q_-^2(\tfrac{x}{\varepsilon}) & \text{on } \partial\{u > 0\} \cap U. \end{cases}$$

Here $|\nabla_a u| = (\nabla u \cdot a(\tfrac{x}{\varepsilon})\nabla u)^{1/2}$.

Some visual stimulation

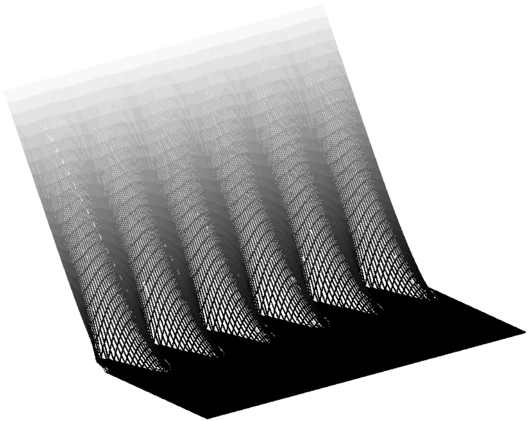

Figure: A particularly nice one-phase minimizer in periodic media might look like this.

Free boundary regularity

Much is known on the regularity theory for minimizers of the homogeneous one-phase functional

$$\mathcal{J}_0(u; U) = \int_U |\nabla u|^2 + q_+^2 \mathbf{1}_{\{u>0\}} + q_-^2 \mathbf{1}_{\{u\leq 0\}} \, dx.$$

Like for minimal surfaces, regularity in low dimensions or flat solutions, singularities occur in higher dimensions for the one-phase problem.

Almost minimality

Definition

Say that u is an (R_0, β)-almost minimizer of $\mathcal{J}_0$ in a domain U if, for any $B_r \subset U$, and any $v \in u + H_0^1(B_r)$

$$\mathcal{J}_0(u; B_r) \leq \mathcal{J}_0(v; B_r) + (r/R_0)^\beta |B_r|.$$

Almost minimizers don't directly solve a PDE, but they generally turn out to have similar regularity theory to minimizers (better than just general PDE solutions). This generality is useful in constrained optimization and shape optimization problems.

Almost minimality

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$$\mathcal{J}_0(u; B_r) \leq \mathcal{J}_0(v; B_r) + (r/R_0)^\beta |B_r|.$$

Homogenization gives a kind of reversed almost minimality property: if u minimizes $\mathcal{J}_\varepsilon$ over $u + H^1_0(B_r)$

$$\mathcal{J}_0(\bar{u}_\varepsilon; B_r) \leq \mathcal{J}_0(v; B_r) + C(\varepsilon/r)^\gamma |B_r| \ \text{ for all } \ v \in u + H^1_0(B_r)$$

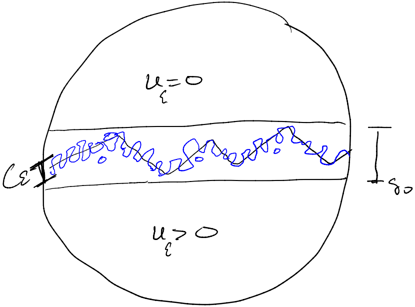
where $\mathcal{J}_0$ is the homogenized functional associated with $\mathcal{J}_\varepsilon$.

Free boundary regularity: heterogeneous medium – one-phase case

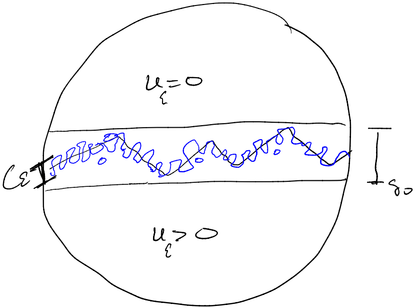
We will exploit ideas from the almost minimizer theory and from quantitative homogenization theory. For the periodic medium functional

$$\mathcal{J}_\varepsilon(u;U) = \int_U a(\tfrac{x}{\varepsilon})\nabla u \cdot \nabla u + Q(\tfrac{x}{\varepsilon})^2 \mathbf{1}_{\{u>0\}}\, dx.$$

we show a flat implies regular result:



Free boundary regularity: heterogeneous medium – one-phase case



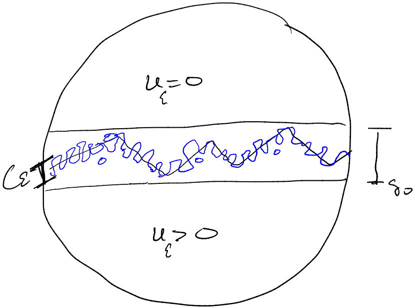
Theorem (F., Ars Inveniendi Analytica '23)

There is $\delta_0 > 0$ sufficiently small so that if u_ε minimizes $\mathcal{J}_\varepsilon$ in B_1 and is δ_0-flat in B_1, i.e. if

$$(x_d - \delta_0)_+ \leq u_\varepsilon(x) \leq (x_d + \delta_0)_+ \quad \text{in} \quad B_1$$

then $\{u_\varepsilon > 0\}$ is distance $C\varepsilon$ from a 1-Lipschitz subgraph in $B_{1/2}$.

Free boundary regularity: heterogeneous medium – one-phase case



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then $\{u_\varepsilon > 0\}$ is distance $C_\eta \varepsilon$ from a η-Lipschitz subgraph in $B_{1/2}$.

Two-phase problem

The two-phase problem

$$\begin{cases} -\Delta u = 0 & \text{in } \{|u| > 0\} \cap U \\ |\nabla u_+|^2 - |\nabla u_-|^2 = q_+^2 - q_-^2 & \text{on } \partial\{u > 0\} \cap U \end{cases}$$

two-plane solutions denoted using

$$\Phi_\alpha(t) = \sqrt{q_+^2 - q_-^2 + \alpha^2}\max\{t, 0\} + \alpha \min\{t, 0\}$$

leading to solutions

$$\Phi_\alpha(x \cdot e) \quad \text{for all} \quad \alpha \geq 0,\ e \in S^{d-1}.$$

Two-phase problem

Two-plane solution

$$\Phi_\alpha(x_d) = \sqrt{1+\alpha^2}\,\max\{x_d, 0\} + \alpha\min\{x_d, 0\}$$

- Unlike in the one-phase case (where $\alpha = 0$) the slope at the free boundary may be arbitrarily large. This makes the Lipschitz estimate a significant challenge. Most approaches in the literature go via the Alt-Caffarelli-Friedman (ACF) monotonicity formula.
- On the other hand, again by the ACF formula, there are no singular two-phase free boundary points. And the only global homogeneous minimizers with a nontrivial negative phase are two-plane solutions.

Alt-Caffarelli-Friedman '84

Results on the two-phase problem

We have not been able to adapt monotonicity formula techniques to the homogenization problem. Instead we follow an idea of De Silva and Savin ('19) proving flat-implies-smooth first, and then inheriting regularity from the *a*-harmonic replacement in the large slope case to prove a Lipschitz estimate.

Theorem (Abedin and F., preprint on arXiv May 2025)

Suppose that u minimizes $\mathcal{J}_\varepsilon$ over $u + H^1_0(B_1)$ then

$$\|\nabla u\|_{\underline{L}^2(B_r)} \leq C(1 + \|\nabla u\|_{\underline{L}^2(B_1)}) \text{ for all } \varepsilon \leq r \leq 1.$$

Here $\|f\|_{\underline{L}^2(B_r)} := (\frac{1}{|B_r|}\int_{B_r} |f|^2 \, dx)^{1/2}$ is the averaged L^2 norm.

Homogenization and large scale regularity: background

Periodic homogenization

Divergence form elliptic equations with periodic coefficients behave well at large scales, this is a central idea of **homogenization**. For example the solution of a Dirichlet problem in a large domain

$$\begin{cases} -\nabla \cdot (a(\frac{x}{\varepsilon})\nabla u_\varepsilon) = 0 & \text{in } U \\ u_\varepsilon = g(x) & \text{on } \partial U \end{cases}$$

is well approximated by a certain constant coefficient *homogenized* equation

$$\begin{cases} -\nabla \cdot (\bar{a}\nabla \bar{u}) = 0 & \text{in } U \\ \bar{u} = g(x) & \text{on } \partial U \end{cases}$$

when $\varepsilon \ll 1$, U is a fixed regular domain, and g is a fixed regular boundary condition.

Corrector problem

The effective matrix $\bar{a}$ is determined by a *corrector problem*, for each $p \in \mathbb{R}^d$ there is a corrector $\chi_p(x)$ which is a mean zero periodic function solving

$$-\nabla \cdot (a(x)(p + \nabla \chi_p(x))) = 0 \text{ in } \mathbb{R}^d.$$

Essentially the corrector "corrects" the linear function so that $p \cdot x + \chi_p(x)$ is a-harmonic. Note that χ_p is actually linear in p.

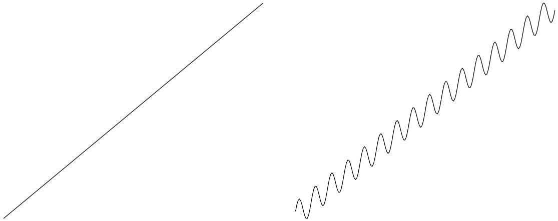


Figure: Left: linear function $p \cdot x$, right: corrected linear function $p \cdot x + \varepsilon \chi_p(x/\varepsilon)$

Effective matrix

The *effective* or *homogenized* coefficient matrix $\bar{a}$ is determined by the relation

$$\bar{a}p = \langle a(p + \nabla \chi_p) \rangle$$

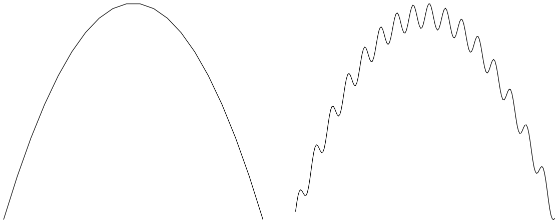
for a vector $p \in \mathbb{R}^d$. Assume that $\bar{a} = \mathrm{id}$ for the rest of the talk for convenience.

Asymptotic expansion and large scale regularity

Formally speaking if u_ε is a_ε-harmonic in a ball $B_1(0)$ (for example) it has an asymptotic expansion

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon \chi_{\nabla \bar{u}(x)}(\tfrac{x}{\varepsilon}) + O(\varepsilon^2)$$

where $\bar{u}$ is the $\bar{a}$-harmonic replacement in $B_1(0)$.



Asymptotic expansion and large scale regularity

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon \chi_{\nabla \bar{u}(x)}(\tfrac{x}{\varepsilon}) + O(\varepsilon^2)$$

and so also expect

$$\nabla u_\varepsilon(x) = \nabla \bar{u}(x) + \nabla \chi_{\nabla \bar{u}(x)}(\tfrac{x}{\varepsilon}) + O(\varepsilon).$$

So for a_ε-harmonic functions the best estimate allowed by the formal expansion is Lipschitz

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \leq C \|\nabla u_\varepsilon\|_{L^2(B_1)}$$

and this does hold in periodic media by work of Avellaneda and Lin ('87-'91).

Large scale regularity

More recently Armstrong and Smart ('16) gave a more quantitative proof, similar to the Campanato iteration approach to Schauder theory, which has had wide application in random homogenization.

- Prove a (sub-optimal) quantitative homogenization homogenization result with algebraic rate.
- Establish a $C^{1,\beta}$-type improvement of flatness by inheriting the regularity of the homogenized problem similar to

$$\frac{1}{\mu r} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{B_{\mu r}(0)} (u_\varepsilon - p \cdot x) \leq \mu^\beta \frac{1}{r} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{B_r(0)} (u_\varepsilon - p \cdot x) + C \left(\frac{\varepsilon}{r}\right)^\alpha .$$

The estimate gives $C^{1,\gamma}$ regularity at intermediate scales and Lipschitz regularity all the way down to scale ε.

Homogenization and large scale regularity: minimizers of the one-phase energy

Γ-limit

Returning to the one-phase problem we might expect a Γ-convergence result for the functionals

$$\mathcal{J}_\varepsilon(u;U) = \int_U a(\tfrac{x}{\varepsilon})\nabla u \cdot \nabla u + Q(\tfrac{x}{\varepsilon})^2 \mathbf{1}_{\{u>0\}} \ dx$$

to a homogenized / effective functional

$$\mathcal{J}_0(u;U) = \int_U |\nabla u|^2 + \langle Q^2 \rangle \mathbf{1}_{\{u>0\}} \ dx.$$

Note that Γ convergence gives information about minimizers in a "big enough" energy well, but not about all local minimizers.

Recall we assume $\bar{a} = \mathrm{id}$ for convenience.

Quantitative Γ limit

$$\mathcal{J}_\varepsilon(u;U) = \int_U a(\tfrac{x}{\varepsilon})\nabla u \cdot \nabla u + Q(\tfrac{x}{\varepsilon})^2 \mathbf{1}_{\{u>0\}}\ dx$$

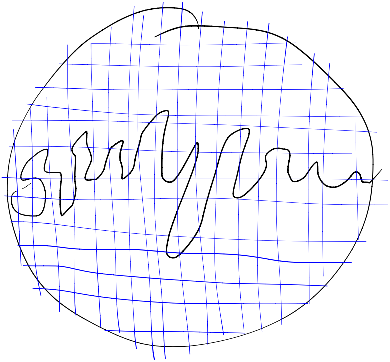


Figure: Need to show that there is not too much energy / period squares crossing the free boundary or the fixed boundary.

Quantitative Γ limit

Ingredients to prove the quantitative homogenization, both are "rough" regularity estimates that only need uniform ellipticity of the coefficients no averaging.

- Energy bound in a strip near the free boundary / perimeter bound
- Up to the (fixed) boundary $W^{1,p}$ regularity for some $p > 2$ (Caffarelli-Peral-type approach to Meyers' estimate)

Then we get a quantitative Γ-convergence estimate similar to

$$\mathcal{J}_0(\bar{u}_\varepsilon; B_r) \leq \mathcal{J}_0(v; B_r) + C(\varepsilon/r)^\gamma |B_r| \ \text{ for all } \ v \in u + H_0^1(B_r).$$

Here $\bar{u}_\varepsilon$ is an explicit regularization of u_ε (mollification with a cutoff near the free boundary). Note the similar form to almost minimality.

Quantitative homogenization and almost minimality

If u_ε is an (R_0, β)-almost minimizer for $\mathcal{J}_\varepsilon$ in U then we can show something similar: for all $B_r \subset U$ and $\varepsilon \leq r \leq R_0$, $v \in u_\varepsilon + H_0^1(B_r)$, and

$$\mathcal{J}_0(\bar{u}_\varepsilon; B_r) \leq \mathcal{J}_0(v; B_r) + C[(r/R_0)^\beta + (\varepsilon/r)^\gamma]|B_r|.$$

Since the error is algebraic it is summable over geometric sequences of scales between ε and R_0.

Improvement of flatness for almost minimizers

Lemma (De Silva and Savin ('20))

Let u satisfy

$$\|\nabla u\|_{L^\infty(B_1)} \leq L \ \text{ and } \ J_0(u; B_1) \leq J_0(v; B_1) + \sigma \ \text{ for all } \ v \in u + H_0^1(B_1).$$

For any $0 < \alpha < 1$ there exist constants $\bar{\delta}, \eta > 0$ and $C \geq 1$ depending on L and α so that if $0 \in \partial\{u > 0\}$ and

$$|u(x) - (x_d)_+| \leq \delta \ \text{ in } \ B_1$$

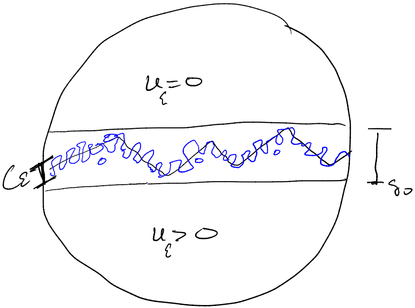
with $0 < \delta + \sigma^{\frac{1}{d+4}} \leq \bar{\delta}$, then there is $\nu \in S^{d-1}$ with $|\nu - e_d| \leq C(\delta + \sigma^{\frac{1}{d+4}})$

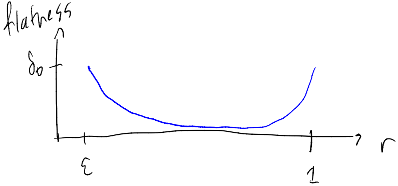
$$|u(x) - (x \cdot \nu)_+| \leq \eta^{1+\alpha}(\delta + \sigma^{\frac{1}{d+4}}) \ \text{ in } \ B_\eta(0).$$

Improvement of flatness for almost minimizers

Proof is based on De Silva's ('11) partial Harnack compactness argument for flat implies $C^{1,\alpha}$ interior estimates. **Key step**: show that approximate minimizers satisfy an approximate viscosity solution property. We generalized this to the two-phase case (Abedin and F., preprint, '25).

Free boundary regularity: heterogeneous medium





Theorem (F., Ars Inveniendi Analytica '23)

There is $\delta_0 > 0$ sufficiently small so that if u_ε minimizes $\mathcal{J}_\varepsilon$ in B_1 and is δ-flat in B_1, i.e. if

$$(x_d - \delta_0)_+ \leq u_\varepsilon(x) \leq (x_d + \delta_0)_+ \quad \text{in} \quad B_1$$

then $\{u_\varepsilon > 0\}$ is distance $C\varepsilon$ from a 1-Lipschitz subgraph in $B_{1/2}$.

A Liouville property

Corollary (F., Ars Inveniendi Analytica '23)

If the only minimals of $\mathcal{J}_0$ on all of $\mathbb{R}^d$ are half-plane solutions (i.e. as is known in $d \leq 4$) and u minimizes $\mathcal{J}_1$ on $\mathbb{R}^d$ with respect to compact perturbations, $0 \in \partial\{u > 0\}$, then there is $\nu_* \in S^{d-1}$ so that

$$\frac{1}{r} \sup_{B_r} |u(x) - (\nu_* \cdot x)_+| \leq C \frac{1}{r^{\frac{1}{d+4}}} \quad \text{for all} \quad r \geq 1.$$

A Liouville property (two-phase)

Corollary (Abedin and F., preprint)

If u minimizes $\mathcal{J}_1$ on $\mathbb{R}^d$ with respect to compact perturbations, $0 \in \partial\{u > 0\}$*, and*

$$\liminf_{r\to\infty} \|\nabla u\|_{\underline{L}^2(B_r)} < +\infty, \quad \textit{and} \quad \inf_{x\in\mathbb{R}^d} \liminf_{r\to\infty} \frac{u(rx)}{r} < 0.$$

Then there is $\nu_* \in S^{d-1}$ *and* $\alpha_* > 0$ *so that*

$$\frac{1}{r}\sup_{B_r} |u(x) - \Phi_\alpha(\nu_* \cdot x)| \leq C\frac{1}{r^\omega} \quad \textit{for all} \quad r \geq 1.$$

Minimizers in the whole space

The problem is to describe solutions of the Euler equation

$$\sum_{i=1}^{n} \partial_{x_i} F_{p_i}(x, u, u_x) = F_u(x, u, u_x)$$

with linear growth $u = O(|x|)$. One may expect that there exists a vector $\alpha \in \mathbb{R}^n$ such that $u - (\alpha, x)$ is bounded in $\mathbb{R}^n$. It turns out that such a statement is false even for $n = 1$. However, we conjecture that it is true for minimal solutions, i.e. functions $u \in H^1_{\text{loc}}$ satisfying

$$\text{(1.11)} \qquad \int_{\mathbb{R}^n} (F(x, u + \varphi, u_x + \varphi_x) - F(x, u, u_x))\, dx \geq 0$$

for all $\varphi \in C^1_{\text{comp}}(\mathbb{R}^n)$. As a matter of fact, in [10] the first author proved such statements for minimals without self-intersections, i.e. minimals u for which for any $j \in \mathbb{Z}^n$, $j_0 \in \mathbb{Z}$ the function $u(x+j) - j_0 - u(x)$ has a fixed sign or vanishes identically.

Figure: Conjecture from Moser and Struwe ('92) on minimals of variational problems on the torus

One phase vs two phase

Two phase

1. Hölder estimate
2. Approximate viscosity solution property when $\alpha > 0$
3. Improvement of flatness in when $\alpha > 0$
4. Lipschitz estimate

One phase

1. Lipschitz estimate
2. Non-degeneracy of positive phase
3. Approximate viscosity solution property
4. Improvement of flatness

Sketch of the two-phase Lipschitz estimate I

Follows an idea from a paper of De Silva and Savin (IMRN, 2019).
Let u be a $\mathcal{J}$-minimizer.

- Let v be the a-harmonic replacement of u in B_R

$$\|\nabla u - \nabla v\|_{\underline{L}^2(B_R)} \leq C.$$

- Dichotomy based on relative size of ξ " $:=$ " $\nabla v(0)$ and $\|\nabla u\|_{\underline{L}^2(B_R)}$:
  - Case (i): If $|\xi| \leq c\|\nabla u\|_{\underline{L}^2(B_R)}$ then

$$\|\nabla u\|_{\underline{L}^2(B_{\eta R})} \leq \frac{1}{2}\|\nabla u\|_{\underline{L}^2(B_R)}.$$

  - Case (ii): If $c\|\nabla u\|_{\underline{L}^2(B_R)} \leq |\xi|$, then the Avellaneda-Lin $C^{1,\beta}$ estimate implies v and hence u are flat with respect to a (corrected) plane. Then $C^{1,\alpha}$ improvement of flatness gives

$$\|\nabla u\|_{\underline{L}^2(B_r)} \leq C_0(1 + \|\nabla u\|_{\underline{L}^2(B_R)}) \text{ for all } \varepsilon \leq r \leq R.$$

A Shape Optimization Problem

Principal Dirichlet eigenvalue shape optimization

Principal Dirichlet eigenvalue for a domain $\Omega \subset \mathbb{R}^n$

$$\lambda_1(\Omega, a_\varepsilon) = \inf\{\int_\Omega a(\tfrac{x}{\varepsilon})\nabla v \cdot \nabla v \; dx : \; v \in H_0^1(\Omega), \; \|v\|_{L^2(\Omega)} = 1\}.$$

Classical Faber-Krahn inequality says

$$\lambda_1(\Omega, \mathrm{id}) \geq \lambda_1(B, \mathrm{id}) \;\text{ for any ball with }\; |B| = |\Omega|.$$

Can we say something about volume constrained minimizers of $\lambda_1(\Omega, a_\varepsilon)$ when $\varepsilon > 0$ is small?

Asymptotic expansion

One might conjecture an asymptotic expansion in ε

$$\inf_{|V|=1} \lambda_1(V, a_\varepsilon) = \lambda_1(B, \mathrm{id}) + L_1\varepsilon + L_2\varepsilon^2 + \cdots \quad \text{where} \quad |B| = 1.$$

Recall we assume that $\bar{a} = \mathrm{id}$ for simplicity. (Note: I do not have any prediction for what L_1 might be, or even if it should be nonzero)

The obvious thing to try

Say that Ω_ε is a domain optimizer

$$\lambda_1(\Omega_\varepsilon, a_\varepsilon) = \inf_{|V|=1} \lambda_1(V, a_\varepsilon).$$

Then from the Faber-Krahn inequality

$$\lambda_1(B, \mathrm{id}) \leq \lambda_1(\Omega_\varepsilon, \mathrm{id}).$$

and so

$$\lambda_1(B, \mathrm{id}) - \lambda_1(\Omega_\varepsilon, a_\varepsilon) \leq \lambda_1(\Omega_\varepsilon, \mathrm{id}) - \lambda_1(\Omega_\varepsilon, a_\varepsilon).$$

Rate of convergence of eigenvalues is well-studied topic in homogenization, but it needs some domain regularity! Kenig, Lin and Shen ('12) show (almost) optimal convergence rate $O(\varepsilon|\log\varepsilon|^{\frac{1}{2}+})$ in Lipschitz domains.

Relaxing the constraint

The hard constraint minimization problem

$$\text{find } \Omega_\varepsilon \text{ satisfying } \ \lambda_1(\Omega_\varepsilon, a_\varepsilon) = \inf_{|V|=1} \lambda_1(V, a_\varepsilon)$$

may often be relaxed to a minimization problem for an *augmented functional* (think of Lagrange multipliers)

$$A_\mu(\Omega, a_\varepsilon) = \lambda_1(\Omega, a_\varepsilon) + \mu|\Omega|$$

and for the connection with the Alt-Caffarelli energy finally note that

$$A_\mu(\Omega, a_\varepsilon) = \inf_{\substack{v \in H_0^1(\Omega), \\ \|v\|_{L^2(\Omega)}=1}} \{ \int_{\mathbb{R}^d} a(\tfrac{x}{\varepsilon}) \nabla v \cdot \nabla v + \mu \mathbf{1}_\Omega \ dx \}.$$

Quantitative homogenization for the optimal eigenvalue

What we can prove implementing the large scale regularity theory described before:

Theorem (F., CPAM '23)

For $p > d + 4$ (not optimal)

$$|\inf_{|V|=1} \lambda_1(V, a_\varepsilon) - \lambda_1(B, \mathrm{id})| \leq C\varepsilon |\log(2 + \varepsilon^{-1})|^p$$

where $C \geq 1$ depends on d, the uniform ellipticity, and $\|\nabla a\|_\infty$.

Can also derive $O(\varepsilon^{1/2}|\log(2 + \varepsilon^{-1})|^{p/2})$ convergence of the optimizing domains Ω_ε in L^1.

Sorry for a change of convention

At this point I want to change scales and instead of fixing volume 1 and sending $\varepsilon \to 0$, I want to fix the length scale of a to be 1 and send the volume $|\Omega| = m$ to infinity. That is we will now consider

$$\inf_{|V|=m} \lambda_1(V, a).$$

Note that then $\varepsilon \sim m^{-\frac{1}{d}}$ and the principal eigenvalue scales like $\lambda_1 \sim m^{-\frac{2}{d}}$.

Outline of the proof: augmented functional

We will rely on a relationship with the *augmented functional*

$$A_\mu(\Omega, a) = \lambda_1(\Omega, a) + \mu|\Omega|.$$

This soft constraint problem is easier to deal with.

- Augmented minimizers are almost minimizers of an Alt-Caffarelli-type functional $\implies$ Lipschitz estimate, non-degeneracy, boundary density estimates.
- This gives sufficient domain regularity for a *sub-optimal* convergence estimate to a ball.
- Convergence to ball $\implies$ large scale flatness.
- Large scale flat implies minimizing domain is Lipschitz (above unit scale).
- For Lipschitz domain we get optimal convergence of $\lambda_1(\Omega, a) - \lambda_1(\Omega, \mathrm{id})$ (up to logarithms) by work of Kenig, Lin and Shen.

Outline of the proof: hard constraint

Now we turn to hard constraint minimization

$$\min\{\lambda_1(V, a) : |V| = m\}.$$

This is a key issue encountered in the shape optimization literature

- Briançon and Lamboley ('09) showed an almost minimality property for shape optimizers of the first Dirichlet eigenvalue with a domain constraint and then used the Alt and Caffarelli regularity strategy to obtain domain regularity. Many other results since.

The issue with previous results in the literature in our case is the almost minimality is not quantitative.

Outline of the proof: hard constraint

Now we turn to hard constraint minimization

$$\min\{\lambda_1(V, a) : |V| = m\}.$$

We no longer can prove regularity directly.

- A_μ minimizers Ω_μ create a monotone multi-valued map $\mathrm{Vol}(\mu)$ by $\mu \mapsto |\Omega_\mu|$. The map can have jump discontinuities where some volumes are missed.
- Dilation / convexity argument at the jump discontinuities of Vol shows that for each Ω_* volume constrained minimizer

  $$m^{\frac{2}{d}}(A_{\mu_*}(\Omega_*, a) - \inf A_{\mu_*}(\cdot, a)) \leq Cm^{-1/d}|\log(2+m)|^{1/2+}.$$

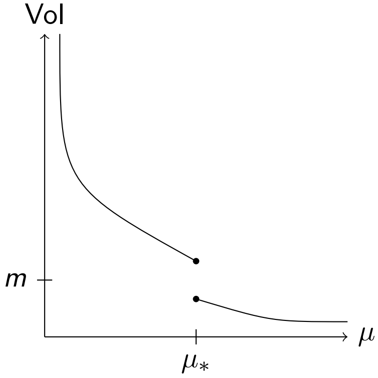
  It is key here that the top and bottom volumes at a jump discontinuity are attained by regular A_μ minimizers.
- Selection principle to find a nearby A_{μ_*} minimizer with nearby principal eigenvalue. (Inspired by Brasco, De Philippis, and Velichkov ('15))

Missed volumes

- Dilation / convexity argument at the jump discontinuities of Vol shows that for each Ω_* volume constrained minimizer

$$m^{\frac{2}{d}}(A_{\mu_*}(\Omega_*, a) - \inf A_{\mu_*}(\cdot, a)) \leq Cm^{-1/d}|\log(2+m)|^{1/2+}.$$

It is key here that the top and bottom volumes at a jump discontinuity are attained by regular A_μ minimizers.



Penalized functional

- Selection principle to find a nearby A_{μ_*} minimizer with nearby principal eigenvalue. (Inspired by Brasco, De Philippis, and Velichkov ('15))

Use a penalized functional

$$B(\Omega, \Omega_*, a) = \lambda_1(\Omega, a) + \int_{\Omega \Delta \Omega_*} \omega\left(\frac{d(x, \partial\Omega_*)}{m^{\frac{1}{d}}}\right) \, dx \qquad (1)$$

Ideally you would want the L^1 difference $|\Omega \Delta \Omega_*|$, but need a little continuity from the modulus ω to get the domain regularity theory to work. Can lose as little as a poly-logarithmic factor by taking a Dini modulus.

Open questions / future directions

- Stronger Liouville theorem, say in $d = 2$: are minimizers in the whole space distance $O(1)$ from a plane?
- Convergence rate of the domains in shape optimization problem seems likely to be suboptimal at $O(\varepsilon^{1/2})$.
- Can we develop a similar large scale regularity theory in any other interface homogenization problems? Capillary problem on heterogeneous surface looks most promising. Minimal surfaces / paths in random environments are much more challenging.

Thanks for your attention!

Dilation / convexity argument

Here consider just the (homogeneous) Laplacian case. For any domain U and any $t > 0$

$$\lambda_1(tU) = t^{-2}\lambda_1(U).$$

Suppose that U_1 and U_2 are A_μ minimizers with $|U_2| > |U_1|$ and U_* is a volume constrained minimizer with $|U_1| < |U_*| < |U_2|$. Call $t_* = (|U_*|/|U_1|)^{1/d}$ and $T = (|U_2|/|U_1|)^{1/d}$ so that $t_* \in (1, T)$. Now, since U is a volume constrained minimizer,

$$\begin{aligned} A_\mu(U_*) &= \lambda_1(U_*) + \mu|U_*| \\ &\leq \lambda_1(tU_1) + \mu|t_* U_1| \\ &= t_*^{-2}\lambda_1(U_1) + \mu t_*^d |U_1| \end{aligned}$$

Now define the convex function on $t \in \mathbb{R}_+$

$$f(t) := t^{-2}\lambda_1(U_1) + \mu t^d |U_1|$$

Dilation / convexity argument

$$f(t) := t^{-2}\lambda_1(U_1) + \mu t^d |U_1|.$$

And we continue, using convexity of f,

$$\begin{aligned}
A_\mu(U_*) &\le f(t_*) \\
&< \max\{f(1), f(T)\} \\
&= \max\{A_\mu(U_1), T^{-2}\lambda_1(U_1) + \mu|U_2|\} \\
&\le \max\{A_\mu(U_1), T^{-2}\lambda_1(T^{-1}U_2) + \mu|U_2|\} \\
&= \max\{A_\mu(U_1), A_\mu(U_2)\} \\
&= \inf A_\mu(V).
\end{aligned}$$

Improvement of flatness: two-phase case key lemma I

Suppose that $v_0, v_1 \in H^1(U) \cap C(U)$ with $v_0 \le v_1$ and $v_0 = v_1 \ge 0$ on ∂U. Call $\Omega_j = \{v_j > 0\} \cap U$.

(i) If v_1 satisfies $\Delta v_1 \ge \mu > 0$ in the H^1 weak sense in $\{v_1 > (v_0)_+\}$ then

$$\int_{\overline{\Omega}_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx$$
$$\ge \int_{\Omega_1 \setminus \overline{\Omega}_0} |\nabla v_1|^2 \, dx + 2\mu \int_{\Omega_1} (v_1 - (v_0)_+) \, dx.$$

(ii) If v_0 satisfies $\Delta v_0 \le -\mu < 0$ in the H^1 weak sense in $\{v_1 > (v_0)_+\}$ then

$$\int_{\overline{\Omega}_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx$$
$$\le \int_{\Omega_1 \setminus \overline{\Omega}_0} |\nabla v_0|^2 \, dx - 2\mu \int_{\Omega_1} (v_1 - (v_0)_+) \, dx$$