Convexity and comparison principle for a singular Bernoulli one-phase problem

Will Feldman

The University of Utah

Anisotropic one-phase problem

We consider an exterior problem for the one-phase Bernoulli problem

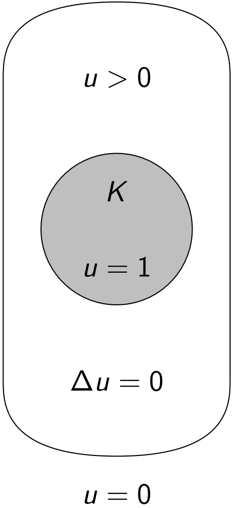
$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \setminus K, \\ |\nabla u| = Q(n_x) & \text{on } \partial\{u > 0\}, \\ u = 1 & \text{on } K. \end{cases} \tag{Ext}$$

Here K is a compact, convex region, n_x is the inner unit normal to $\partial\{u > 0\}$ at x, and $Q : S^{d-1} \to (0, \infty)$ is bounded above and below and upper semi-continuous

$$Q(n) \geq \limsup_{n' \to n} Q(n').$$

An example

Caffarelli and Lee observed that discontinuities of Q can cause facets in the free boundary.



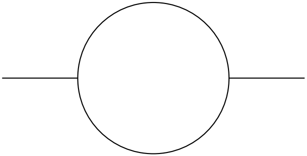


Figure: Plot of $Q(n)$ over $n \in S^1$

Caffarelli and Lee, CPDE 2007

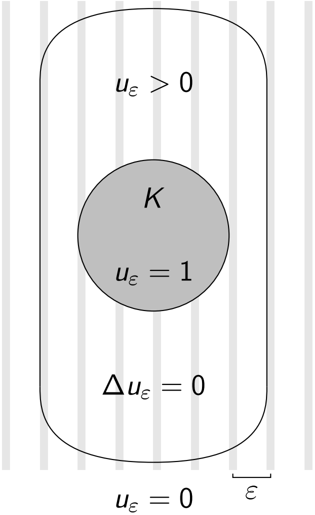
Where does this PDE come from?

Homogenization in a laminar medium

Let 1-periodic $q : \mathbb{R} \to [\frac{1}{2}, 2]$, with $\langle q^2 \rangle = 1$, $\max q = 2$. Consider the minimal supersolutions u_ε of the PDE

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \{u > 0\} \setminus K, \\ |\nabla u_\varepsilon| = q(\frac{x_1}{\varepsilon}) & \text{on } \partial\{u > 0\}, \\ u_\varepsilon = 1 & \text{on } K. \end{cases} \tag{1}$$

Note: There are more than one solution of this PDE, the energy minimizing solution converges to a ball instead.



Caffarelli and Lee (CPDE, 2007), Kim (CPDE, 2008)
Feldman (ARMA, 2021)

A discrete free boundary problem

Consider the minimal supersolution $u_N : \mathbb{Z}^d \to [0, N]$ of the discrete one-phase Bernoulli problem

$$\begin{cases} \Delta_{\mathbb{Z}^d} u_N = 0 & \text{in } \{u_N > 0\} \setminus NK, \\ \Delta_{\mathbb{Z}^d} u_N \leq 1 & \text{on } \partial_{out}\{u_N > 0\} \setminus NK, \end{cases} \quad \text{with } \; u_N = N \; \text{ on } NK.$$

Rescalings

$$\bar{u}_N(x) = N^{-1} u_N(Nx)$$

converge to the minimal supersolution of

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \{\bar{u} > 0\} \\ |\nabla \bar{u}| = \bar{Q}(n_x) & \text{on } \partial\{\bar{u} > 0\} \end{cases} \quad \text{with } \; \bar{u}_N = 1 \; \text{ on } K.$$

F. and Smart (ARMA, '19)

Scaling limit: $d = 2$

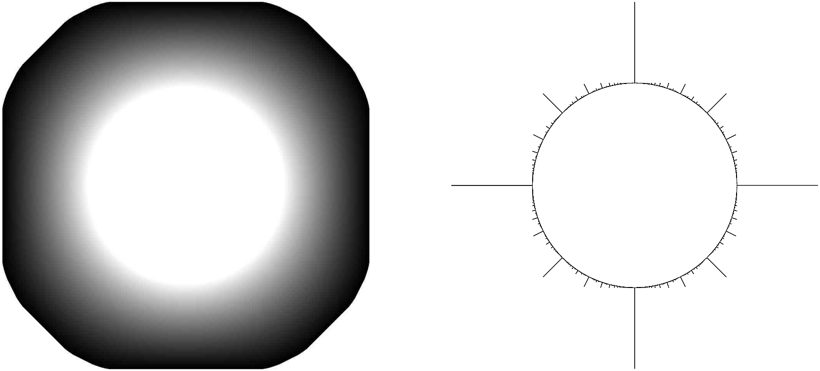


Figure: Left: grayscale map of u_N, Right: plot of $\bar{Q}$ over S^1

F. and Smart (ARMA, '19)

Scaling limit: $d = 3$

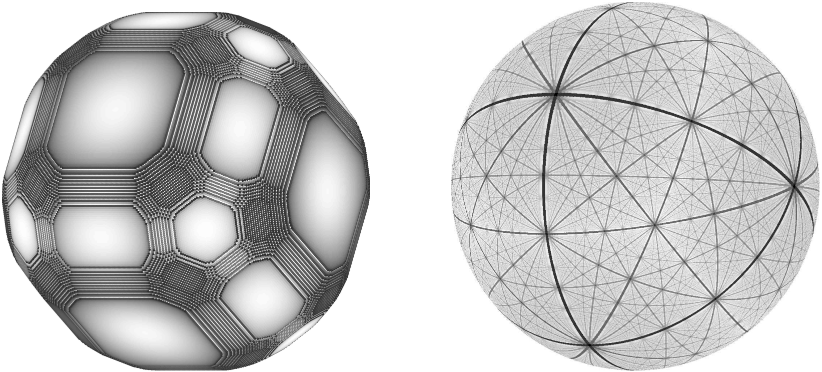


Figure: Left: grayscale map of $\Delta_{\mathbb{Z}^d} u_N$ on $\partial_{\text{out}}\{u > 0\}$, Right: grayscale plot of $\bar{Q}$ over S^2

F. and Smart (ARMA, '19)

Structure of the effective PDE

Theorem (F. and Smart, ARMA '19)

Define $S : 2\pi\mathbb{T}^d \to \mathbb{R}$ *by* $S(\theta) = -\log(1 + \frac{1}{d}\sum_{j=1}^d \cos\theta_j)$*, and let* $\hat{S} : \mathbb{Z}^d \to \mathbb{C}$ *be the corresponding Fourier transform. Then* $\hat{S}$ *is real and positive on* $\mathbb{Z}^d$ *and for all* $e \in S^{d-1}$,

$$\bar{Q}(e) = \frac{1}{\sqrt{2d}} \exp\left(\frac{1}{2} \sum_{k \in \mathbb{Z}^d :\ k \cdot e = 0} \hat{S}(k)\right).$$

Proving these scaling limits

A very big picture sketch of the proof of the scaling limit:

- Use "correctors" and perturbed test function method to show a viscosity solution property for all subsequential limits. Discontinuities of the limit PDE limit the type of test functions which can be "corrected".
- Apply an appropriate comparison principle for the limit problem.

Note: There is no way to directly show that the subsequential limits are *minimal*. Instead we show a "local" viscosity property and then prove a comparison uniqueness result to identify the minimal supersolution.

A convex comparison principle

Viscosity solutions: supersolution

Definition

A supersolution of (Ext) is a nonnegative function $u \in C(\mathbb{R}^d)$ that is compactly supported, $u \geq 1$ on K, is harmonic in $\{u > 0\} \setminus K$, and whenever $\varphi \in C^\infty(U)$, U open, $\Delta\varphi > 0$ and $\nabla\varphi \neq 0$ in U, touches u from below at $x \in \partial\{u > 0\} \cap U$ then

$$|\nabla\varphi(x)| \leq Q\left(\frac{\nabla\varphi}{|\nabla\varphi|}(x)\right).$$

Viscosity solutions: weak subsolution

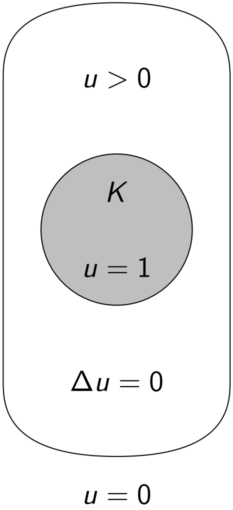
We say that $\varphi \in C^\infty(U)$ is *one-dimensional in U* if it is of the form $\varphi(x) = f(x \cdot p)$ in U for some $f \in C^\infty(\mathbb{R})$ and $p \in \mathbb{R}^d$, $|p| = 1$.

Definition

A *weak subsolution* of (Ext) is a nonnegative function $u \in C(\mathbb{R}^d)$ that is compactly supported, satisfies $u \leq 1$ on K, is harmonic in $\{u > 0\} \setminus K$, and such that, whenever U is an open neighborhood and one-dimensional φ with $\Delta\varphi < 0$ touches u from above in $\overline{\{u > 0\}}$ at $x \in \partial\{u > 0\} \cap U$ with strict ordering $u < \varphi$ on $\overline{\{u > 0\}} \cap \partial U$, then

$$|\nabla\varphi(x)| \geq Q(p).$$

Viscosity solutions: weak subsolution intuition



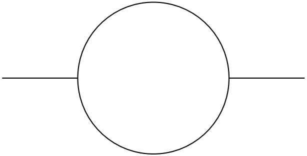


Figure: Plot of $Q(n)$ over $n \in S^1$

Viscosity solutions: weak subsolution intuition

The moral of the story is: if u is a weak subsolution with $\{u > 0\}$ convex, $n \in S^{d-1}$, and

$$F_n = \partial\{u > 0\} \cap \{x \cdot n = \min_{\{u>0\}}(x \cdot n)\}$$

is a facet of $\partial\{u > 0\}$ then

$$\max_{F_n} |\nabla u(x)| \geq Q(n).$$

Convex comparison

Theorem

*A supersolution of (*Ext*) is minimal if and only if it satisfies the weak subsolution property. In other words, a supersolution that is also a weak subsolution is the unique minimal supersolution. Furthermore the minimal supersolution has convex superlevel sets.*

Analogous previous results for classical Bernoulli problem:

- Beurling ('57), Schaeffer ('75), Acker ('78), 2-d conformal mapping techniques.
- Hamilton ('82), Nash-Moser implicit function theorem idea, also 2-d.
- Henrot and Shahgholian ('97,'02), and Bianchini ('12,'22). Maximum principle techniques in all dimensions.

. .

F. and Smart (ARMA, 2019)
new proof in F. and Požár (preprint, 2024)

Quasiconcavity of the capacity gradient on a facet

Let $K \subset\subset V$ both convex and compact, and V inner regular. Let v be the capacity potential

$$\begin{cases} \Delta v = 0 & \text{in } V \setminus K, \\ v = 1 & \text{on } K, \\ v = 0 & \text{on } \mathbb{R}^d \setminus V. \end{cases} \tag{2}$$

The super-level sets $\{v > t\}$ are convex for all $t \in (0,1)$ and let Λ be a tangent hyperplane to ∂V. Set $F = \partial V \cap \Lambda$. Then $1/|\nabla v|$ is convex on F.

Caffarelli & Spruck

Outline of comparison principle proofs

Background of the comparison proof:

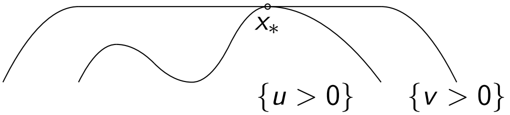
- (Framework) We will aim to prove comparison between a convex (sub / supersolution) and a general (super / subsolution).
- (Regularization) We can always assume that the supersolution has outer regular positivity set, and the subsolution has inner regular positivity set by using inf / sup convolutions.
- (Dilations) We can create a touching point by dilation of the supersolution $v \mapsto v(\lambda x)$ for $\lambda < 1$. This also, conveniently, makes the supersolution strict. Uses K star-shaped.

Convex supersolution touches weak subsolution from above

Lemma

*Assume Q is upper semicontinuous. Suppose that v is supersolution of (*Ext*) with $\{v > 0\}$ convex, and u is a weak subsolution of (*Ext*). Then $v \geq u$.*

This direction is easy: build a one-dimensional test function touching u from above at x_* with a too small slope.



Supersolution touches convex weak subsolution from above

As discussed before the weak subsolution condition implies that

$$\max_{F_n} |\nabla u(x)| \geq Q(n).$$

But it is not immediate to get a contradiction from this:

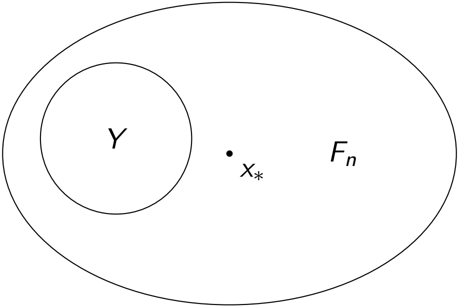


Figure: View of a 2-d boundary facet F_n. Subsolution definition is saturated at x_* which is not on the contact set Y with the supersolution.

Using convexity of $1/|\nabla u|$

Lemma

Suppose that u is a weak subsolution with convex and inner regular positive set $\{u > 0\}$ and $n \in S^{d-1}$ then

$$\min_{F_n} |\nabla u| \geq \liminf_{n' \to n} Q(n').$$

Recall that an *exposed point* of a convex set X is a point x_0 in ∂X so that $\{x_0\} = \{x \in X : x \cdot n = \min_{y \in X} y \cdot n\}$.

Use that the weak subsolution condition at an exposed point $x_0 \in \partial\{u > 0\}$ gives

$$|\nabla u(x_0)| = \max_{F_{n_{x_0}}} |\nabla u(x)| \geq Q(n_{x_0})$$

plus Straszewicz's Theorem –every extreme point is a limit of exposed points – and convexity of $1/|\nabla u|$ to conclude.

Naive continuous approximation works!

Lemma

Suppose that u is a weak subsolution with convex and inner regular positive set $\{u > 0\}$ and $n \in S^{d-1}$ **and Q is continuous** *then*

$$\min_{F_n} |\nabla u| \geq Q(n).$$

Now for upper semicontinuous Q can just naively approximate from above by $Q^j \searrow Q$ with Q^j continuous. Corresponding minimal supersolutions $u^j \nearrow u^\infty \leq u$, since u^j are supersolutions of Q equation. But u^∞ is a convex supersolution and so easy direction of comparison (convex supersolution and general weak subsolution) implies $u^\infty \geq u$.

Regularity

Convexity implies free boundary C^1 regularity

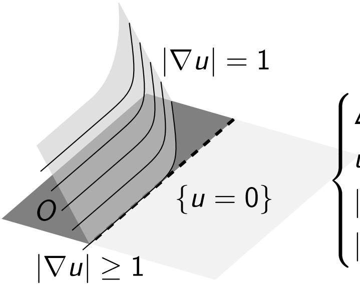
Solution u of exterior problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \setminus K, \\ |\nabla u| = Q(n_x) & \text{on } \partial\{u > 0\}, \\ u = 1 & \text{on } K. \end{cases} \tag{Ext}$$

is quasiconvex. Can also show that $\{u > 0\}$ is a C^1 domain. If $x_0 \in \partial\{u > 0\}$ had a nontrivial supporting cone, blowing-up contradicts Lipschitz regularity and non-degeneracy of u, since positive harmonic functions in a (non-half-space) convex cone with zero Dirichlet data are homogeneous of degree $\alpha > 1$.

One-phase problem with an obstacle

A function $u \in C(\overline{U})$ is a solution of the Bernoulli problem in U with obstacle O from above if



$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ u = 0 & \text{in } O^{\complement} \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap O \\ |\nabla u| \geq 1 & \text{on } \Lambda := \partial\{u > 0\} \cap \partial O. \end{cases}$$

Theorem (Chang-Lara and Savin, Contemp. Math. '19)

Let u solves the Bernoulli obstacle problem in B_1 and $0 \in \partial'\Lambda$, O a C^2 obstacle. Suppose $0 \in \partial'\Lambda$ and u is ε_0-flat in B_1 then $u \in C^{1,1/2}(B_{1/2} \cap \overline{\{u > 0\}})$.

One-phase problem with an obstacle

The regularity is proved by a (rigorous) asymptotic expansion involving the thin obstacle problem. If u solves in B_1 the flattened obstacle problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ u = 0 & \text{in } \{x_d \geq 0\} \end{cases} \quad \text{and} \quad \begin{cases} |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap \{x_d > 0\} \\ |\nabla u| \geq 1 & \text{on } \partial\{u > 0\} \cap \{x_d = 0\}. \end{cases}$$

and is ε-flat (sufficiently small universal)

$$(x_d - \varepsilon)_+ \leq u(x) \leq (x_d + \varepsilon)_+$$

then there is w solving the *thin obstacle* or *Signorini* problem in B_1 with $\operatorname{osc}_{B_1} w \leq 1$ such that

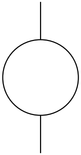
$$u(x) = (x_d - \varepsilon w(x) + o(\varepsilon))_+.$$

approach to free boundary regularity by De Silva '11

Simple anisotropy

Let's return to the anistropic Bernoulli problem and consider now a simple type of discontinuity

$$Q(n) = \begin{cases} 2 & n \in \{\pm e_2\} \\ 1 & \text{else} \end{cases}$$



Solution u of the exterior problem is convex and therefore it solves the Bernoulli obstacle problem with obstacle

$$O := \{|x_2| < \max_{x \in \{u>0\}} |x_2|\}.$$

Simple anisotropy

Solution u of the exterior problem is convex and therefore it solves the Bernoulli obstacle problem with obstacle

$$O := \{|x_2| < \max_{x \in \{u>0\}} |x_2|\}.$$

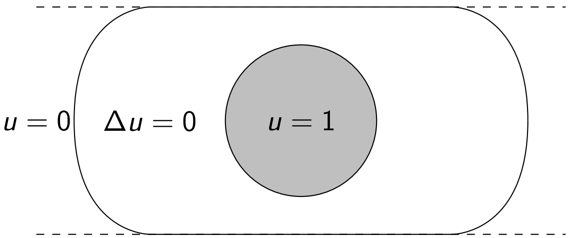


Figure: O is region between dashed lines. Drawn rotated by $\pi/2$.

Unfortunately this idea is limited to the case of convex data, and to *simple discontinuities* where $\lim_{n' \to n} Q(n')$ exists but is not equal to $Q(n)$.

Regularity: non-convex case

Without convexity regularity problem seems a lot harder. Consider the simple discontinuity at $n = e_d$, as earlier. Can show a flat asymptotic expansion: if u monotone in the e_d direction and flat

$$(x_d - \varepsilon)_+ \leq u(x) \leq (x_d + \varepsilon)_+ \ \text{ in } \ B_1$$

then

$$u(x) = (x_d + \varepsilon w(x) + o(\varepsilon))_+ \ \text{ in } \ B_{1/2}$$

where w solves the *gradient degenerate Neumann problem*

$$\Delta w = 0 \ \text{ in } \ B_1^+, \ \text{ and } \ \min\{\partial_d w, |\nabla' w|\} = 0 \ \text{ on } \ B_1'.$$

forthcoming work with PhD student Zhonggan Huang

Regularity: non-convex case

The *gradient degenerate Neumann problem*

$$\Delta w = 0 \ \text{ in } \ B_1^+, \ \text{ and } \ \min\{\partial_d w, |\nabla' w|\} = 0 \ \text{ on } \ B_1'. \qquad (3)$$

The *contact set* is $\Lambda := \{x \in B_1' : \ \partial_d w > 0\}$.

Theorem (F. and Huang, forthcoming)

In $d = 2$, if w solves (3) then $w \in C^{1,1/2}_{loc}(B_1^+)$. In $d \geq 3$ if $\# w(\Lambda \cap B_1) < +\infty$ then $w \in C^{1,1/2}_{loc}(B_1^+)$.

forthcoming work with PhD student Zhonggan Huang

Questions

- Regularity theory beyond point discontinuities. For example in $d = 3$: $Q(n) = 1 + \mathbf{1}_{n \cdot e_3 = 0}$ "Saturn with its rings" type Q.
- Comparison principle without convexity beyond simple discontinuities (limits exist at every n but may not agree with the value).
- Volume constrained problem: does there exist a (unique modulo translation?) quasiconvex solution (supersolution and weak subsolution) of

$$\begin{cases} -\Delta u = \lambda & \text{in } \{u > 0\}, \\ |\nabla u| = Q(n_x) & \text{on } \partial\{u > 0\}, \\ \int_{\mathbb{R}^d} u = 1. \end{cases} \tag{Vol}$$

No apparent variational structure to help.

Thank you for your attention!