Mean curvature flow with positive random forcing in 2-d

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Thanks to the Friends of the IAS for their support.

Forced mean curvature flow

Region S_t evolving by normal velocity with planar initial data $S_0 = \{e \cdot x \leq 0\}$

$$V_n = -\kappa + c(x) + F.$$

Here κ is mean curvature, $c(x)$ is inhomogeneous environment, constant F is large scale external driving force (e.g. pressure, contact angle, or magnetic field).

Level set form

$$u_t = \mathrm{tr}((I - \tfrac{\nabla u \otimes \nabla u}{|\nabla u|^2})D^2 u) + (c(x) + F)|\nabla u| \quad \text{with} \quad u(x,0) = e \cdot x.$$

Model for

- Domain boundaries in magnetic materials
- Flow in porous media
- Contact line motion

Pinning interval

Expectation: there is a pinning interval $[F_*(e), F^*(e)]$

$$F^*(e) = \inf\{F : \lim_{t\to\infty} \inf_{x\in\partial S_t} \frac{x \cdot e}{t} > 0\}.$$

Front has positive speed outside of the pinning interval.

Question (Homogenization)

Does the propagating interface stay flat for $F > F^*(e)$ and propagate with some asymptotic speed $\bar{c}(e)$?

In general answer is no in all $d \geq 2$, so we need to refine the question.

Propagation as a flat front

Periodic media:

- (Lions and Souganidis, 2005) Lipschitz estimates and existence of correctors under the coercivity condition

$$\inf_{\mathbb{R}^d}[(c(x)+F)^2-(d-1)|Dc|]>0.$$

  Lipschitz estimates can fail without this condition.

- (Dirr, Karali and Yip, 2008) If c smooth and small C^2-norm then initially flat fronts stay flat and propagate with an asymptotic speed. Note no coercivity condition.
- (Caffarelli and Monneau, 2014) Counter-example in $d \geq 3$, homogenization in $d=2$ without a Lipschitz estimate with weak coercivity

$$\inf_{\mathbb{R}^d}(c+F)>0.$$

Propagation as a flat front

Random media:

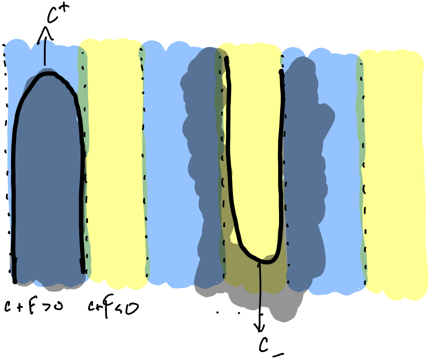
- (Armstrong and Cardaliaguet, 2015) Homogenization in $d \geq 2$ with the Lions-Souganidis coercivity condition, finite range dependence random field.

Main Result (F., preprint 2019)

Homogenization in $d = 2$ with Caffarelli-Monneau coercivity (i.e. $\inf(c + F) > 0$), finite range dependence random field.

Examples

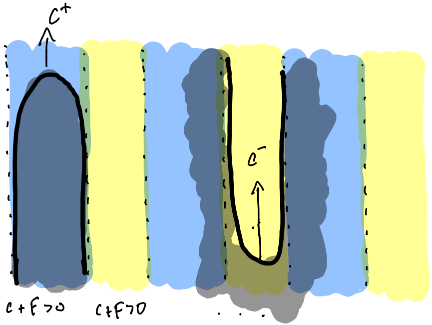
Fingering phenomenon, $c + F$ signed in 2-d:



(Cardaliaguet, Lions and Souganidis, 2007)
(Dirr, Karali and Yip, 2008)

Examples

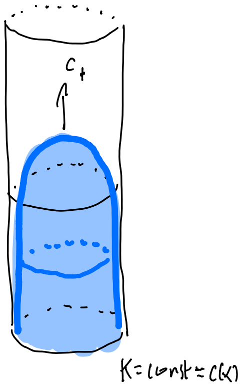
Fingering phenomenon, $c + F$ signed in 2-d:

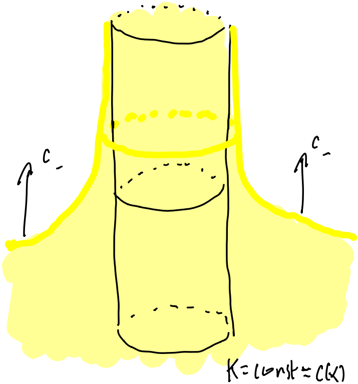


In this example $F > F^*(e_2)$ (implied by Cardaliaguet, Lions and Souganidis, 2007) but homogenization fails.

Examples

Fingering phenomenon, $c + F$ positive in 3-d
(Caffarelli and Monneau, 2014)





Some open questions

Question

In $d = 2$, random or periodic media, does homogenization hold when

$$F > \sup_e F^*(e)?$$

(Still wouldn't be a sharp condition, e.g. see laminar case)

Question

Does failure of homogenization imply the existence of unbounded stationary solutions? A decomposition into travelling waves of various speeds connecting stationary solutions?

- Laminar media (Cesaroni and Novaga, 2013), head and tail speeds (Gao and Kim, 2018).

Arrival time function

Normalize $F = 0$ now.

It is useful to formulate the problem in terms of the arrival time $m(x, S)$, the first time the front started from the set S (often a half-space $S = \{e \cdot x \leq 0\}$) arrives at the point x.

Since $c > 0$ can show that $S_t = \{m(x) \leq t\}$ with m solving

$$\mathrm{tr}((I - \tfrac{\nabla u \otimes \nabla u}{|\nabla u|^2})D^2 m) + c(x)|\nabla m| = 1 \ \text{ in } \mathbb{R}^d \setminus S$$

with boundary data $m(x) = 0$ in S.

Positive random forcing in $d = 2$

In the case $\inf c > 0$ and finite range of dependence, propagating fronts stay "flat"

Theorem (F., preprint)

For every direction e there is a deterministic asymptotic speed $\overline{c}(e)$ so that the arrival time m of the front started from $\{x \cdot e \leq 0\}$ satisfies,

$$\mathbb{P}(|m(re) - \mathbb{E}[m(re)]| > \lambda r^{1/2}) \leq Ce^{-c\lambda^2}$$

and

$$|\mathbb{E}[m(re)] - \frac{1}{\overline{c}(e)} r| \leq Cr^{2/3}.$$

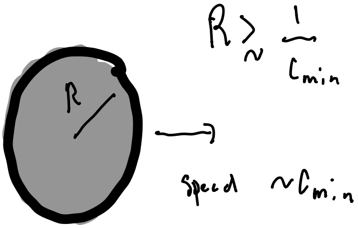
The effective velocity $\overline{c} : S^1 \to (0, \infty)$ is continuous with logarithmic modulus of continuity.

Ideas in the proof

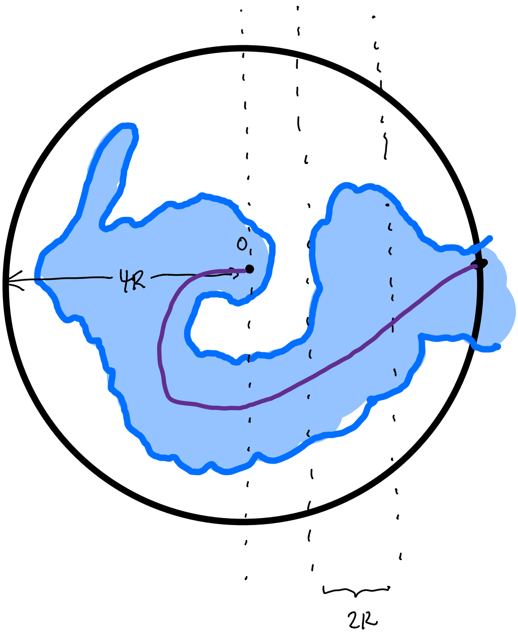
1. Large scale Lipschitz estimate of the arrival time (using weak coercivity / controllability given by the condition $\inf c > 0$)
$$|m(x) - m(y)| \leq C + C|x - y|.$$
2. Martingale decomposition and Azuma's inequality give bound on the variance (Armstrong-Cardaliaguet-Souganidis, Armstrong-Cardaliaguet).
3. Localized uniqueness result without local regularity.

Waiting time / large scale Lipschitz estimate

Caffarelli and Monneau's moving ball barrier



Caffarelli and Monneau's moving ball barrier



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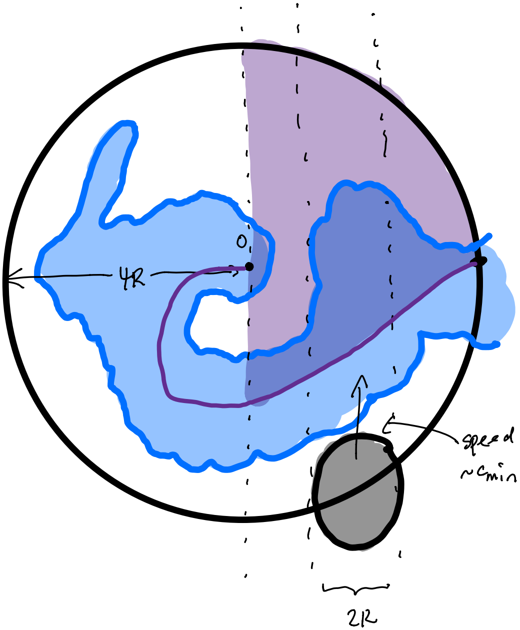


Lipschitz estimate

Above argument shows that, with $R = R(d, \inf c)$,

$$\max_{y \in B_R(x)} m(y) \leq m(x) + C(d, \inf c).$$

Iterate to get large scale Lipschitz.

Localized uniqueness

A little local regularity

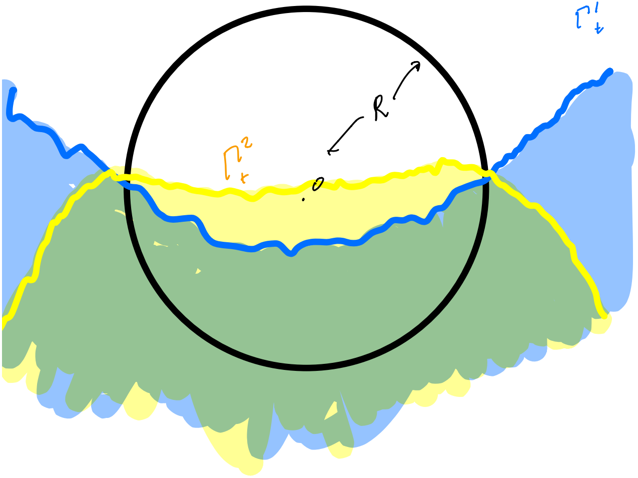
Lemma

Let $S \subset \mathbb{R}^d$ regular (radius $R(\inf c)$ interior ball condition).

$$|m(x,S) - m(y,S)| \leq \frac{2}{c_{\min}} e^{\|\nabla c\|_\infty m(x) \wedge m(y)} |x-y|.$$

(Souganidis, 1985)(Crandall and Lions, 1986)

Localized uniqueness



Previous localized uniqueness results:
(Armstrong and Cardaliaguet, 2015)(Gao and Kim, 2018)

Lemma

Let m^1 and m^2 be solutions of the arrival time problem,

$$-\mathrm{tr}\left[(I - \tfrac{Dm\otimes Dm}{|Dm|^2})D^2 m\right] + c(x)|Dm| = 1 \quad \text{in} \quad \mathbb{R}^2 \setminus S^i$$

with $m^i = 0$ in S^i. Suppose that both S^i are regular, there is $R \geq 1$ such that the ordering holds

$$S^2 \subset S^1 \quad \text{on} \quad B_R(0).$$

Then there exists $C \geq 1$ such that, if $s \geq 1$ and $R \geq \bar{R}(s) = e^{Cs}$,

$$\{m(x, S^2) \leq s - 1\} \subset \{m(x, S^1) \leq s\} \quad \text{on} \quad B_{R-\bar{R}(s)}(0).$$

Now iterate, using waiting time / large scale Lipschitz to regularize at each iteration.

Lemma

Let m^1 and m^2 be solutions of the arrival time problem,

$$-\mathrm{tr}\left[(I - \tfrac{Dm\otimes Dm}{|Dm|^2})D^2 m\right] + c(x)|Dm| = 1 \ \ \textit{in} \ \ \mathbb{R}^2 \setminus S^i$$

with $m^i = 0$ in S^i. Suppose that: m^i both satisfy the large scale Lipschitz estimate, and there is $R \geq 1$ such that the ordering holds

$$S^2 \subset S^1 \ \ \textit{on} \ \ B_R(0).$$

Then, if $s \geq 1$, $n \leq s$, and $R \geq \bar{R}_n(s) = ne^{C\frac{s}{n}}$,

$$\{m^2(x) \leq s - Cn\} \subset \{m^1(x) \leq s\} \ \ \textit{on} \ \ B_{R-\bar{R}_n(s)}(0).$$

Future directions

- Flat fronts down to the pinning interval endpoint?
- Other models e.g. contact angle dynamics ...
- Is the depinning transition sharp? (could depend on the model)

Other interesting issues

Recall

Expectation: there is a pinning interval $[F_*(e), F^*(e)]$

$$F^*(e) = \inf\{F : \liminf_{t\to\infty} \tfrac{u(0,t)}{t} > 0\}.$$

Front has positive speed outside of the pinning interval. Can also define the depinning transition value

$$F^d(e) = \inf\{F : \lim_{t\to\infty} u(0,t) = +\infty \ \text{ a.s}\}$$

Open questions:

1. Are the depinning and positive speed transitions the same?
2. What is the behavior of $\bar{c}(F)$ near the depinning threshold? Conjectured universality $\bar{c}(F) \sim (F - F^*)^\theta$.

Critical transitions

- Single critical transition in special i.i.d. fully discrete model (Bodineau and Teixeira, 2015)
- Sub-ballistic propagation at the pinning interval endpoint in partially discrete model with long range correlations (Dondl and Scheutzow, 2017)

Depinning exponent

- Periodic media (related models) $\bar{c}(F) \sim (F - F^*)^{1/2}$ (Dirr and Yip, 2006)
- Abstract setting $\bar{c}(F) \sim (F - F^*)^{1-\kappa/2}$ where κ effective dimension of medium (Scheel and Tikhomirov, 2017)

Model problem to think of in 1-d: suppose $c : \mathbb{R} \to \mathbb{R}$ has minimum 0 at 0 and

$$\dot{x} = c(x) + F \approx \alpha x^2 + F.$$

Pinning for $F \leq 0$, depinned for $F > 0$ takes time $\sim F^{1/2}$ to pass through neighborhood of 0.