Appendix B
Answers to Selected Problems

The aim here is to provide answers, but not solutions, to all problems that call for answers. However, there are about 20 answers that have been left blank because work is incomplete. Therefore, this document will be updated when missing answers become available. (Version 1.0, August 29, 2010.)

Version 1.1, November 13, 2012: The answer to Problem 4.1(e) has been corrected, thanks to Brent Kerby. Problems 17.13 and 17.14(c) have been solved by John Jungtae Kim.

Chapter 1

1.1 33,203,125.

1.2 (a) 302,500. (b) 103,411.

1.5 ace high: 502,860; king high: 335,580; queen high: 213,180; jack high: 127,500; ten high: 70,380; nine high: 34,680; eight high: 14,280; seven high: 4,080.

1.6 (a) 7,462. (b) straight flush: 10 of 4 each; four of a kind: 156 of 4 each; full house: 156 of 24 each; flush: 1,277 of 4 each; straight: 10 of 1,020 each; three of a kind: 858 of 64 each; two pair: 858 of 144 each; one pair: 2,860 of 384 each; no pair: 1,277 of 1,020 each. (c) A-K-Q-J-7 and A-K-Q-J-6.

1.7 straight flush: 52; flush: 5,096; straight: 13,260; no pair: 1,299,480.

1.9 J-J-A-T-8.

1.10 five of a kind: 6; four of a kind: 150; full house: 300; three of a kind: 1,200; two pair: 1,800; one pair: 3,600; no pair: 720.

1.11 2: 0.251885; 3: 0.508537; 4: 0.200906; 5: 0.0354212; 6: 0.00309421; 7: 0.000152027; 8: 0.394905×10^{-5}; 9: 0.402327×10^{-7}; 10: 0.690305×10^{-10}. 2: 10; 3: 380; 4: 2,610; 5: 7,851; 6: 13,365; 7: 13,896; 8: 8,041; 9: 2,209; 10: 170.

1.12 $\sum_{m=1}^{n}(-1)^{m-1}(n)_m s^m/[m!(ns)_m]$; 0.643065.

1.15 0.00525770.

1.16 0.545584; 0.463673.

1.17 0.461538.

1.18 (b) There are nine others: 111234 144556; 112226 234566; 112233 122346; 112256 125566; 112456 113344; 122255 124456; 122336 123444; 122455 124455; 123456 222444.

1.19 Yes.

1.20 (a) 2: 0; 3: 0; 4: 0.056352; 5: 0.090164; 6: 0.128074; 7: 0.338115; 8: 0.128074; 9: 0.090164; 10: 0.056352; 11: 0.112705; 12: 0. (b) 2: 0.054781; 3: 0.109562; 4: 0.109562; 5: 0.131474; 6: 0.149402; 7: 0; 8: 0.149402; 9: 0.131474; 10: 0.109562; 11: 0; 12: 0.054781.

1.22 0.094758.

1.23 0.109421.

1.27 −0.078704.

1.28 0.464979; −0.070041.

1.29 (a) −0.034029. (b) 3.330551.

1.30 λ; λ.

1.34 1.707107; 1.

1.36 (a) 14.7; 13. (b) 61.217385; 52.

1.37 89.830110; 86.

1.39 10.5; 8.75.

1.41 For two dice, probabilities multiplied by 36 are 1; 2; 3; 4; 5; 6; 5; 4; 3; 2; 1. For three dice, probabilities multiplied by 216 are 1; 3; 6; 10; 15; 21; 25; 27; 27; 25; 21; 15; 10; 6; 3; 1. For four dice, probabilities multiplied by 1,296 are 1; 4; 10; 20; 35; 56; 80; 104; 125; 140; 146; 140; 125; 104; 80; 56; 35; 20; 10; 4; 1. For five dice, probabilities multiplied by 7,776 are 1; 5; 15; 35; 70; 126; 205; 305; 420; 540; 651; 735; 780; 780; 735; 651; 540; 420; 305; 205; 126; 70; 35; 15; 5; 1. For six dice, probabilities multiplied by 46,656 are 1; 6; 21; 56; 126; 252; 456; 756; 1,161; 1,666; 2,247; 2,856; 3,431; 3,906; 4,221; 4,332; 4,221; 3,906; 3,431; 2,856; 2,247; 1,666; 1,161; 756; 456; 252; 126; 56; 21; 6; 1.

1.42 0.467657; 0.484182; 0.048161; −0.016525.

1.45 (b) Other than standard dice, no.

Chapter 2

2.1 (a) 17 : 5 : 5.

2.2 (b) $1/(p_1 + p_2)$. (c) 3.272727.

2.3 $1/p$; $(1-p)/p^2$.

2.4 (a) 0.457558. (b) 6.549070.

2.5 2.407592.

2.6 2.938301; 3.801014.

2.7 8.525510.

2.8 (a) p_1/p_2. (b) 3/6 (resp., 4/6; 5/6; 5/6; 4/6; 3/6).

2.10 (a) λp; λp. (b) Poisson(λp).

2.12 (a) $1-\sum_i(1-p_i)^n+\sum\sum_{i<j}(1-p_i-p_j)^n-\cdots$. (c) $p_m(j)=(j/k)p_{m-1}(j)+[1-(j-1)/k]p_{m-1}(j-1)$. (d) 152.
2.13 multinomial$(n-\sum_i k_i, (\sum_i q_i)^{-1}\boldsymbol{q})$; yes.

Chapter 3

3.7 12.25.
3.8 $3abc/(a+b+c)$; $ab+ac+bc$.
3.10 (a) For $n=j,j+1,\ldots,j+m-1$,

$$M_{n,j}=\sum_{i=0}^{n-j}p_{u_1}^{-1}\cdots p_{u_i}^{-1}1_{\{X_j=u_1,\ldots,X_{j+i-1}=u_i\}}\big(p_{u_{i+1}}^{-1}1_{\{X_{j+i}=u_{i+1}\}}-1\big),$$

while for $n\ge j+m$, $M_{n,j}=M_{n-1,j}$.
3.11 The probability is $(C-D)/(A-B+C-D)$, where

$$A:=\sum_{1\le i\le m:\ (u_{m-i+1},\ldots,u_m)=(u_1,\ldots,u_i)}(p_{u_1}\cdots p_{u_i})^{-1},$$

$$B:=\sum_{1\le i\le m\wedge l:\ (u_{m-i+1},\ldots,u_m)=(v_1,\ldots,v_i)}(p_{v_1}\cdots p_{v_i})^{-1},$$

$$C:=\sum_{1\le i\le l:\ (v_{l-i+1},\ldots,v_l)=(v_1,\ldots,v_i)}(p_{v_1}\cdots p_{v_i})^{-1},$$

$$D:=\sum_{1\le i\le m\wedge l:\ (v_{l-i+1},\ldots,v_l)=(u_1,\ldots,u_i)}(p_{u_1}\cdots p_{u_i})^{-1}.$$

3.13 $p=\beta/(\alpha+\beta)$ is required for martingale property; replace $=$ by $\le$ (resp., $\ge$) for the supermartingale (resp., submartingale) property.
3.14 (c) Consider $\mathrm{P}(X_1=2)=1/3$ and $\mathrm{P}(X_1=-1)=2/3$.
3.15 (b) The requirement is that $p\ln(1-\alpha)+(1-p)\ln(1+\alpha)<0$. (c) Yes.
3.16 Yes.
3.18 $\mu=7/2$. The limit $W\ge 0$ exists a.s. and $\mathrm{E}[W]\le 1$.

Chapter 4

4.1 (a)–(d) are Markov chains. (e) is not.
(a)

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 2 & & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 3 & & & 3/6 & 1/6 & 1/6 & 1/6 \\ 4 & & & & 4/6 & 1/6 & 1/6 \\ 5 & & & & & 5/6 & 1/6 \\ 6 & & & & & & 1 \end{array}$$

(b)

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & 5/6 & 1/6 & & & & \\ 1 & & 5/6 & 1/6 & & & \\ 2 & & & 5/6 & 1/6 & & \\ 3 & & & & 5/6 & 1/6 & \\ \vdots & & & & & & \end{array}$$

(c)

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & 1/6 & 5/6 & & & & \\ 1 & 1/6 & & 5/6 & & & \\ 2 & 1/6 & & & 5/6 & & \\ 3 & 1/6 & & & & 5/6 & \\ \vdots & & & & & & \end{array}$$

(d)

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & \cdots \\ \hline 1 & 1/6 & (5/6)(1/6) & (5/6)^2(1/6) & (5/6)^3(1/6) & \\ 2 & 1 & & & & \\ 3 & & 1 & & & \\ 4 & & & 1 & & \\ 5 & & & & 1 & \\ \vdots & & & & & \end{array}$$

4.3

$$\boldsymbol{P}^n = \frac{1}{p+q}\begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (1-p-q)^n \end{pmatrix}\begin{pmatrix} q & p \\ 1 & -1 \end{pmatrix}$$

so

$$\lim_{n\to\infty} \boldsymbol{P}^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix}.$$

4.4 0.496924.
4.10 (a)

$$P(i,j) = \begin{cases} p & \text{if } j = (2i) \wedge m, \\ q & \text{if } j = 0 \vee (2i-m). \end{cases}$$

(b) For $i = 1, 2, \ldots, m-1$, $Q(i) = pQ((2i) \wedge m) + qQ(0 \vee (2i - m))$. (c) For $i = 1, 2, \ldots, m-1$, $R(i) = 1 + pR((2i) \wedge m) + qR(0 \vee (2i - m))$.

4.12 If $p > \frac{1}{2}$, the stationary distribution is shifted geometric$(1 - q/p)$.

4.16 (a)

$$\boldsymbol{P}_B = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_1 & 1-p_1 & 0 \end{pmatrix}.$$

(d)

$$\boldsymbol{P}_A = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

and $\boldsymbol{P}_C = \frac{1}{2}(\boldsymbol{P}_A + \boldsymbol{P}_B)$. For game A, $\pi_0 = 1/3$. For game C, π_0 is as in (4.200) except that p_0 and p_1 are replaced by $(p+p_0)/2$ and $(p+p_1)/2$. For game A, the limit in (4.201) is -2ε. For game C, it is $\pi_0(p+p_1-1)+(1-\pi_0)(p+p_1-1)$ with π_0 modified as just described.

4.17 (b) It is given recursively by

$$\begin{aligned} \pi(i,0) &= p_{i-1}/[m(1-q)], \quad i = 0, 1, \ldots, m-1, \\ \pi(i,1) &= q_{i-1}\pi(i-1,0), \quad i = 0, 1, \ldots, m-1, \\ \pi(i,2) &= q_{i-1}\pi(i-1,1), \quad i = 0, 1, \ldots, m-1, \\ &\vdots \\ \pi(i,m-1) &= q_{i-1}\pi(i-1,m-2), \quad i = 0, 1, \ldots, m-1, \end{aligned}$$

where $q := q_0 q_1 \cdots q_{m-1}$, $p_{-1} := p_{m-1}$, $q_{-1} := q_{m-1}$, and $\pi(-1,j) := \pi(m-1,j)$.

4.18 X_n has distribution equal to the minimum of a shifted geometric(p) and n. Y_n is geometric(p). Z_n is the sum of independent copies of X_n and Y_n.

4.19 0.703770; 1.671987.

4.21 (a) 21.026239; yes. (b) Let $w(i,j)$ denote the probability that the next roller wins if he lacks i and his opponent lacks j. Then

$$w(i,j) = 1 - \sum_{k=3}^{24\wedge i} p_k w(j, i-k),$$

where $w(i,0) := 0$ for $i \geq 1$ and $\{p_k\}$ is the distribution of the number of points obtained in one roll. So $w(167,167) \approx 0.559116$.

Chapter 5

5.2 (a)

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 55 & 10 \\ 10 & 110 \end{pmatrix} \end{array}$$

has optimal mixed strategy $(20/29, 9/29)$ for both Alex and Olaf, and the value (for Alex) is 1,190/29. (b) Exact side payment is 1,190/29.

5.3 Case $d < a < b$. If $b \leq c$, then column 2 is strictly dominated, hence there is a unique optimal solution. If $b > c$, then row 2 is strictly dominated, hence there is a unique optimal solution. Case $d < a = b$. Regardless of c, the unique optimal strategy for player 1 is $\boldsymbol{p} = (1, 0)$, whereas player 2 has $\boldsymbol{q} = (q_1, q_2)$ optimal provided $q_1 d + (1 - q_1)c \leq a$. All choices will work if $c \leq a$, whereas if $c > a$, it suffices that $1 \geq q_1 \geq (c - a)/(c - d)$. Case $d = a < b$. Regardless of c, the unique optimal strategy for player 2 is $\boldsymbol{q} = (1, 0)$, whereas player 1 has $\boldsymbol{p} = (p_1, p_2)$ optimal provided $p_1 b + (1 - p_1)c \geq a$. All choices will work if $c \geq a$, whereas if $c < a$, it suffices that $1 \geq p_1 \geq (a - c)/(b - c)$. Case $d = a = b$. If $c \geq a$, then every $\boldsymbol{p}$ for player 1 is optimal. If $c < a$, then $\boldsymbol{p} = (1, 0)$ is uniquely optimal. If $c \leq a$, then every $\boldsymbol{q}$ for player 2 is optimal. If $c > a$, then $\boldsymbol{q} = (1, 0)$ is uniquely optimal.

5.4 $\boldsymbol{p}^* = v\boldsymbol{1}\boldsymbol{A}^{-1}$, $(\boldsymbol{q}^*)^\mathsf{T} = v\boldsymbol{A}^{-1}\boldsymbol{1}^\mathsf{T}$, and $v = (\boldsymbol{1}\boldsymbol{A}^{-1}\boldsymbol{1}^\mathsf{T})^{-1}$; yes.

5.5 The optimal mixed strategy is $(4/5, 1/5)$ for both players, and the value for player 1 is $-4/5$.

5.6 (a) $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ are equal to $(a_1^{-1}, \ldots, a_m^{-1})$ divided by the sum of the components, and v is the reciprocal of the sum of the components. (b) There is a saddle point and the value is 0.

5.7 $N = 3$: $(2, 2)$ is a saddle point; the optimal strategy is unique. $N = 4$: $(2, 2)$ is a saddle point; the optimal strategy is nonunique. $N = 6$: $(3, 3)$ is a saddle point; the optimal strategy is nonunique.

5.8 Both strategies are optimal.

5.11 Value is $-3(16n^2 - 304n + 11)/[13(52n - 1)(52n - 2)]$, which is negative if $n \geq 19$.

5.12 See Vanniasegaram (2006).

5.15 (a) Draw with probability 0.545455. (b) Draw with probability 0.006993.

5.16 The solution is the same as with conventional rules.

5.17 There is a saddle point: Player draws to 5, and banker uses DSDS. Value (to player) is -0.012281.

5.18 Lemma 5.1.3 reduces the game to $2^5 \times 2^{18}$, according to a masters project at the University of Utah by Carlos Gamez (August 2010).

5.19 There is a saddle point: Player draws to 5 or less; banker draws to 6 or less if player stands, draws to 5 or less if player draws. Value (to player) is -0.011444.

5.20 There are seven pairs of rows that are negatives of each other. For any such pair, mixing the two rows equally is optimal.

Chapter 6

6.1 $(lL - wW)/[l(L+W)]$; 1/66.
6.2 (a1) 1/18. (a2) 1/36. (b) 1/9. (c) 1/6. (d) 1/9. (e1) 1/6. (e2) 5/36. (f1) 1/6. (f2) 1/9. (g1) 1/6. (g2) 1/8.
6.3 0.034029.
6.4 0.484121.
6.5 (a) 0.023749; 0.028936. (b) No.
6.6 (a) $7/(495 + 330m)$. (b) $7/[495(1+m)]$.
6.8 $(\sum b_j)/\{37[b_{\min} + \sum(b_j - b_{\min})]\}$, where $b_{\min} = \min b_j$.
6.9 0.024862.
6.10 0.032827.
6.11 2 and 161/3; 0.037267.
6.12 (b) If $\mathrm{E}[X_j] \le 0$ for $j = 1, \ldots, d$,

$$\begin{aligned} \mathrm{H}^*(X_1 + \cdots + X_d) &= \sum_{i=1}^d \frac{\mathrm{E}\big[|X_i|\big]}{\mathrm{E}\big[|X_1 + \cdots + X_d|\big]} \mathrm{H}^*(X_i) \\ &\ge \sum_{i=1}^d \frac{\mathrm{E}\big[|X_i|\big]}{\mathrm{E}\big[|X_1|\big] + \cdots + \mathrm{E}\big[|X_d|\big]} \mathrm{H}^*(X_i). \end{aligned}$$

(c) $(-ap+q)/(ap+q)$; 0.052632, 0.034483, 0.014085; 0.032258. (d)

$$\frac{\sqrt{n}}{\sigma}\left(\frac{-(X_1 + \cdots + X_n)}{|X_1| + \cdots + |X_n|} - \mathrm{H}^*(X)\right) \xrightarrow{d} N(0,1),$$

where

$$\sigma^2 := \frac{\mathrm{E}[\{X + \mathrm{H}^*(X)|X|\}^2]}{\big(\mathrm{E}\big[|X|\big]\big)^2}.$$

6.14

$$\mathrm{H}_1(\boldsymbol{B}, \boldsymbol{X}) = \frac{-\{\mathrm{E}[X_1 \mid X_1 \ne 0] + \cdots + \mathrm{E}[X_d \mid X_d \ne 0]\}}{\mathrm{E}[B_1 \mid X_1 \ne 0] + \cdots + \mathrm{E}[B_d \mid X_d \ne 0]}$$

Next,

$$\mathrm{H}_1(\boldsymbol{B}, \boldsymbol{X}) = \sum_{i=1}^d \frac{\mathrm{E}[B_i \mid X_i \ne 0]}{\mathrm{E}[B_1 \mid X_1 \ne 0] + \cdots + \mathrm{E}[B_d \mid X_d \ne 0]} \mathrm{H}(B_i, X_i),$$

so $\mathrm{H}_1(\boldsymbol{B}, \boldsymbol{X})$ is a mixture of $\mathrm{H}(B_i, X_i)$. $H_1 \approx 0.039015$.

6.15

$$\mathrm{H}_2(\boldsymbol{B},\boldsymbol{X}) = \frac{-\mathrm{E}[X_1+\cdots+X_d]}{\mathrm{E}[(B_1+\cdots+B_d)1_{\{\boldsymbol{X}\neq\boldsymbol{0}\}}]},$$

so

$$\begin{aligned}\mathrm{H}_2(\boldsymbol{B},\boldsymbol{X}) &= \sum_{i=1}^d \frac{\mathrm{E}[B_i 1_{\{X_i\neq 0\}}]}{\mathrm{E}[(B_1+\cdots+B_d)1_{\{\boldsymbol{X}\neq\boldsymbol{0}\}}]}\mathrm{H}(B_i,X_i)\\ &\le \sum_{i=1}^d \frac{\mathrm{E}[B_i 1_{\{X_i\neq 0\}}]}{\mathrm{E}[B_1 1_{\{X_1\neq 0\}}]+\cdots+\mathrm{E}[B_d 1_{\{X_d\neq 0\}}]}\mathrm{H}(B_i,X_i).\end{aligned}$$

$\mathrm{H}_2 = 0.0125$.

6.16

$$\begin{aligned}\mathrm{H}_1^*(\boldsymbol{X}) &= \sum_{i=1}^d \frac{\mathrm{E}\big[|X_{N_i,i}|\big]}{\mathrm{E}\big[|X_{N_1,1}+\cdots+X_{N_d,d}|\big]}\mathrm{H}^*(X_i)\\ &\ge \sum_{i=1}^d \frac{\mathrm{E}\big[|X_{N_i,i}|\big]}{\mathrm{E}\big[|X_{N_1,1}|\big]+\cdots+\mathrm{E}\big[|X_{N_d,d}|\big]}\mathrm{H}^*(X_i).\end{aligned}$$

$\mathrm{H}_1^* \approx 0.060163$.

6.17 $n = 35, 36, 67, 71, 72$ only.

6.19 (c) $\mathrm{H}^\beta < \mathrm{H}$ if and only if $\beta > (1-ap/q)^2/[1-(1-ap/q)^2]$; in even-money 38-number roulette, $\beta > 1/99$. $\mathrm{H}^\beta = 0$ if and only if $\beta = q/(ap)-1$; in even-money 38-number roulette, $\beta = 1/9$.

6.20 (a)

$$\mathrm{H}^\beta = \frac{\mathrm{H}(B,X)+\beta(1+\beta)^{-1}\mathrm{E}[X\,1_{\{X<0\}}]/\mathrm{E}[B\,1_{\{X\neq 0\}}]}{\mathrm{E}[X\,1_{\{X>0\}}]/q}.$$

(b) 0.050329.

Chapter 7

7.4 Since $p \neq q$, we have $\mu \neq 0$ and

$$\mathrm{Var}(N) = \frac{\sigma^2\mathrm{E}[N]-\mathrm{E}[S_N^2]+2\mu\mathrm{E}[S_N N]}{\mu^2} - (\mathrm{E}[N])^2.$$

Using $\mathrm{E}[S_N N] = -L\mathrm{P}(S_N=-L)\mathrm{E}[N \mid S_N=-L] + W\mathrm{P}(S_N=W)\mathrm{E}[N \mid S_N=W]$, the formula follows from the results cited.

7.5 $\mathrm{P}(N(-2,2)=2n \mid S_{N(-2,2)}=2) = (2pq)^{n-1}(1-2pq)$; conditioning on $S_{N(-2,2)}=-2$ gives the same expression.

7.8 $(p^3+3p^2q)/(p^3+3p^2q+pq^2+q^3)$.

7.10 In (7.85), interchange p and q as well as L and W.

7.11 $[L(-2)^{W+1}+(W+1)(-2)^{-L}-L(-2)^{W}-W(-2)^{-L}-1]/[W(-2)^{W+1}-(W+1)(-2)^{W}+L(-2)^{W+1}+(W+1)(-2)^{-L}-L(-2)^{W}-W(-2)^{-L}]$.

7.12 With $\rho := q/p$, define

$$P := \frac{1}{3}\sqrt[3]{(7+27\rho)/2+\sqrt{8+(7+27\rho)^2/4}},$$
$$Q := -\frac{1}{3}\sqrt[3]{-(7+27\rho)/2+\sqrt{8+(7+27\rho)^2/4}}.$$

Then $\lambda_1 = P+Q-1/3$,

$$\lambda_2 = -\frac{1}{2}(P+Q)-\frac{1}{3}+\frac{1}{2}\sqrt{3}(P-Q)i,$$

and $\lambda_3 = \bar{\lambda}_2$. The required probability is $a_0+a_1+a_2+a_3$, where

$$\begin{aligned} a_0+a_1\lambda_1^{-L}+a_2\lambda_2^{-L}+a_3\lambda_3^{-L} &= 0,\\ a_0+a_1\lambda_1^{W}+a_2\lambda_2^{W}+a_3\lambda_3^{W} &= 1,\\ a_0+a_1\lambda_1^{W+1}+a_2\lambda_2^{W+1}+a_3\lambda_3^{W+1} &= 1,\\ a_0+a_1\lambda_1^{W+2}+a_2\lambda_2^{W+2}+a_3\lambda_3^{W+2} &= 1. \end{aligned}$$

7.14 0.489024.

7.15 $2(p^2+3pq+q^2)/(p^3+3p^2q+pq^2+q^3)$.

7.16 $[W(W+1)/2+L(W+1)(L+W+1)/2-LW(L+W)/2-(-2)^{-L}W(W+1)/2-(-2)^{W}L(W+1)(L+W+1)/2+(-2)^{W+1}LW(L+W)/2]/[W(-2)^{W+1}-(W+1)(-2)^{W}+L(-2)^{W+1}+(W+1)(-2)^{-L}-L(-2)^{W}-W(-2)^{-L}]$.

7.18 30.224945.

7.21 See Table B.1.

7.24

$$\frac{L^2+(W^2-(L^*)^2)P}{\mathrm{E}[X^2]} \le \mathrm{E}[N] \le \frac{(L^*)^2+((W^*)^2-L^2)P}{\mathrm{E}[X^2]},$$

and replace P by P_- or P_+, depending on the sign of the difference in front of it.

Chapter 8

8.1 $1-2p$.

8.2 Let N be the time of the first loss. Then

Table B.1 Accuracy of second-moment approximation (Problem 7.21).

L	approx.	exact	rel. error	L	approx.	exact	rel. error
1	.005832	.005813	.003342	51	.386588	.384950	.004256
2	.011723	.011684	.003359	52	.396325	.394637	.004276
3	.017673	.017614	.003376	53	.406160	.404423	.004296
4	.023684	.023604	.003393	54	.416095	.414307	.004316
5	.029755	.029654	.003410	55	.426130	.424290	.004335
6	.035887	.035765	.003427	56	.436266	.434374	.004355
7	.042082	.041937	.003444	57	.446505	.444560	.004375
8	.048339	.048172	.003461	58	.456847	.454848	.004395
9	.054659	.054470	.003479	59	.467294	.465239	.004415
10	.061043	.060831	.003496	60	.477846	.475736	.004436
11	.067492	.067256	.003514	61	.488505	.486337	.004456
12	.074006	.073745	.003531	62	.499271	.497046	.004476
13	.080585	.080300	.003549	63	.510146	.507863	.004496
14	.087231	.086921	.003566	64	.521131	.518788	.004517
15	.093944	.093609	.003584	65	.532227	.529823	.004537
16	.100725	.100364	.003601	66	.543435	.540970	.004558
17	.107574	.107186	.003619	67	.554756	.552228	.004578
18	.114493	.114078	.003637	68	.566192	.563600	.004599
19	.121481	.121039	.003655	69	.577743	.575086	.004619
20	.128541	.128070	.003673	70	.589411	.586689	.004640
21	.135671	.135172	.003691	71	.601197	.598407	.004661
22	.142873	.142345	.003709	72	.613101	.610244	.004682
23	.150148	.149591	.003727	73	.625126	.622200	.004703
24	.157497	.156909	.003745	74	.637273	.634277	.004723
25	.164920	.164302	.003763	75	.649542	.646475	.004744
26	.172418	.171768	.003782	76	.661935	.658795	.004765
27	.179991	.179310	.003800	77	.674453	.671240	.004787
28	.187641	.186928	.003818	78	.687098	.683810	.004808
29	.195369	.194622	.003837	79	.699870	.696507	.004829
30	.203174	.202394	.003855	80	.712772	.709331	.004850
31	.211058	.210244	.003874	81	.725803	.722285	.004871
32	.219022	.218173	.003892	82	.738967	.735369	.004893
33	.227067	.226182	.003911	83	.752263	.748584	.004914
34	.235192	.234272	.003930	84	.765694	.761933	.004935
35	.243400	.242443	.003948	85	.779260	.775416	.004957
36	.251690	.250696	.003967	86	.792963	.789035	.004978
37	.260065	.259032	.003986	87	.806805	.802790	.005001
38	.268524	.267453	.004005	88	.820787	.816687	.005020
39	.277068	.275958	.004024	89	.834909	.830717	.005046
40	.285699	.284548	.004043	90	.849175	.844900	.005059
41	.294417	.293225	.004062	91	.863584	.859204	.005099
42	.303222	.301990	.004081	92	.878140	.873696	.005086
43	.312117	.310843	.004101	93	.892842	.888244	.005176
44	.321102	.319785	.004120	94	.907692	.903119	.005064
45	.330177	.328816	.004139	95	.922693	.917783	.005350
46	.339345	.337939	.004159	96	.937845	.933319	.004850
47	.348604	.347154	.004178	97	.953150	.947554	.005906
48	.357958	.356461	.004197	98	.968610	.964861	.003885
49	.367405	.365862	.004217	99	.984226	.976457	.007956
50	.376948	.375358	.004237				

$$F_n = \begin{cases} F_0 + (2^n - 1) & \text{if } n \le N-1, \\ F_0 - 1 & \text{if } n \ge N. \end{cases}$$

If there is no house limit, then $\mathrm{P}(F_N = F_0 - 1) = 1$. Let $M \ge 1$ be the house limit and put $m := 1 + \lfloor \log_2 M \rfloor$. The gambler wins $2^m - 1$ units with probability p^m and loses 1 unit with probability $1 - p^m$. Hence the expected cumulative profit is $\mathrm{E}[F_{N\wedge m} - F_0] = (2p)^m - 1$. Finally, the expected number of coups is $\mathrm{E}[N \wedge m] = (1 - p^m)/(1 - p)$ and the expected total amount bet is $\mathrm{E}[2^{N\wedge m} - 1] = (1 - (2p)^m)/(1 - 2p)$ if $p \ne \frac{1}{2}$; $= m$ if $p = \frac{1}{2}$.

8.3 Let N be the time of the first win. Then

$$F_n = \begin{cases} F_0 - (2^{n+1} - (n+2)) & \text{if } n \le N-1, \\ F_0 + N & \text{if } n \ge N. \end{cases}$$

If there is no house limit, then $\mathrm{P}(F_N = F_0 + N) = 1$. Let $M \ge 1$ be the house limit and put $m_1 := \lfloor \beta^{-1}(F_0) \rfloor$, where $\beta(x) := 2^{x+1} - (x+2)$. $m_2 := \lfloor \log_2(M+1) \rfloor$, and $m := m_1 \wedge m_2$. The gambler achieves his goal (winning $N \le m$ units) with probability $1 - q^m$ and loses $2^{m+1} - (m+2)$ units with probability q^m. We find that the expected cumulative profit is $\mathrm{E}[F_{N\wedge m} - F_0] = 2[1 - (2q)^m] - [(1 - q^m)/(1 - q)](1 - 2q)$. Finally, the expected number of coups is $\mathrm{E}[N \wedge m] = (1 - q^m)/(1 - q)$ and the expected total amount bet is $\mathrm{E}[2^{(N\wedge m)+1} - (N \wedge m) - 2] = 2[1 - (2q)^m]/(1 - 2q) - (1 - q^m)/(1 - q)$ if $p \ne \frac{1}{2}$; $= 2(m - 1 + 2^{-m})$ if $p = \frac{1}{2}$.

8.6 $(2p - q)^{-1}[j_0 + (1 - \lambda_2)^{-1}(1 - \lambda_2^{-j_0})]$, where λ_2 is as in (7.83).

8.8 (d) For all $m \ge 0$,

$$\begin{aligned}
P_1(N = 3m+1) &= a_m p^{m+1} q^{2m}, \\
P_2(N = 3m+3) &= a_{m+1} p^{m+2} q^{2m+1}, \\
P_3(N = 3m+2) &= a_{m+1} p^{m+2} q^{2m}, \\
P_4(N = 3m+4) &= (a_{m+2} - a_{m+1}) p^{m+3} q^{2m+1}, \\
P_5(N = 3m+3) &= (a_{m+2} - 2a_{m+1}) p^{m+3} q^{2m}, \\
P_6(N = 3m+5) &= (a_{m+3} - 3a_{m+2}) p^{m+4} q^{2m+1},
\end{aligned}$$

$$\begin{aligned}
P_1(N = 3m+2) &= b_m p^{m+1} q^{2m+1}, \\
P_2(N = 3m+1) &= b_m p^{m+2} q^{2m}, \\
P_3(N = 3m+3) &= b_{m+1} p^{m+2} q^{2m+1}, \\
P_4(N = 3m+2) &= (b_{m+1} - b_m) p^{m+3} q^{2m}, \\
P_5(N = 3m+4) &= (b_{m+2} - 2b_{m+1}) p^{m+3} q^{2m+1}, \\
P_6(N = 3m+3) &= (b_{m+2} - 3b_{m+1}) p^{m+4} q^{2m},
\end{aligned}$$

$P_1(N = 3m+3) = 0$, $P_2(N = 3m+2) = 0$, $P_3(N = 3m+4) = 0$, $P_4(N = 3m+3) = 0$, $P_5(N = 3m+5) = 0$, $P_6(N = 3m+4) = 0$.

8.9 $-E[F_n \mid N \geq n+1] = S_0\{P_{j_0}(N \geq n+1)^{-1} - 1\}$, where S_0 is the sum of the terms of the initial list.

8.11 When $M = 3$,

$$Q(1,1) = \left(p + \frac{p^2q^2 + p^3(2q^4+q^5) + p^4(q^5+q^6+2q^7+q^8)}{1-pq} + \frac{p^4(q^5+q^6) + p^6(q^8+q^9+q^{10})}{(1-pq)^2}\right) \cdot \left(1 - pq - p^2q^3 - p^3q^6 - \frac{p^3(q^4+q^5) + p^5(q^7+q^8+q^9)}{1-pq}\right)^{-1}.$$

8.12 The system is as in (8.59), except that the conditions $j \leq i \leq I(j)$ and $1 \leq j \leq M$ are replaced by $j \leq i \leq M - j + 1$ and $1 \leq j \leq \lfloor (M+1)/2 \rfloor$.

8.14

8.15 The win goal is 2(number of wins). We have $N < \infty$ a.s. if and only if $p \geq \frac{1}{2}$, and $\mathrm{E}[N] < \infty$ if and only if $p > \frac{1}{2}$. Finally, the required F_0 when the house limit is M units is $2\binom{M+1}{2}$.

8.16 The transition matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p_1 & 0 & 1-p_1 & 0 & 0 & 0 & \\ 0 & p_2 & 0 & 1-p_2 & 0 & 0 & \\ 0 & p_3 & 0 & 0 & 1-p_3 & 0 & \\ 0 & p_4 & 0 & 0 & 0 & 1-p_4 & \\ \vdots & & & & & & \end{pmatrix} \end{array},$$

where $p_1 := u(1, K)$ and $p_j := u((j-1)K/2, K)$ for $j \geq 2$, in the notation of the chapter. The probability of interest is $p_1/\{1-(1-p_1)[1-\prod_{j\geq 2}(1-p_j)]\}$, which is 1 if and only if $p \geq \frac{1}{2}$.

8.17 $p > \frac{1}{2}$.

8.18

8.19 (b)

8.20 (a) The transition probabilities are of the form

$$P((i,j,k),(i',j',k')) = \begin{cases} p & \text{if } (i',j',k') = (i-j, j \wedge (i-j), k - 1_{\{i-jk \leq j\}} 1_{\{k \geq 1\}}) \\ q & \text{if } (i',j',k') = (i+j, j + 1_{\{k=K-1\}}, (k+1)1_{\{k<K-1\}}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$P((0,0,0),(i',j',k')) = \begin{cases} 1 & \text{if } (i',j',k') = (1,1,0) \\ 0 & \text{otherwise.} \end{cases}$$

(b) Win goal is 1. $N < \infty$ a.s. and $\mathrm{E}[N] < \infty$ if and only if $p \geq \frac{1}{2}$. The required F_0 when the house limit is M is $\binom{M+1}{2}K$.

8.21

8.22

Chapter 9

9.1 $Q_{p/(p+q)}(f)$, where Q_p is the function Q of Section 9.1.

9.7 (b) If $u_k = 1$, then $E(.u_1 \cdots u_k) = \sum_{j=0}^{k-1} p_{u_1} \cdots p_{u_j}$, where $p_0 := p$ and $p_1 := q = 1-p$. (c) Let $k \to \infty$ in part (b). (d) If $u_k = 1$, then $E(.u_1 \cdots u_k) = 2(1-2^{-k})$; if $.u_1 u_2 \cdots$ is nonterminating, then $E(.u_1 u_2 \cdots) = 2$.

9.8 (b) If $u_k = 1$, then $B(.u_1 \cdots u_k) = \sum_{j=0}^{k-1} h(.u_{j+1} \cdots u_k) p_{u_1} \cdots p_{u_j}$, where $h(f) := f \wedge (1-f)$, $p_0 := p$, and $p_1 := q = 1-p$. (c) Let $k \to \infty$ in part (b). (d) $B(f) = (Q(f) - f)/(2p-1)$.

9.9 $(1 - \sqrt{1-4rp})/(2r)$; 0.493057.

Chapter 10

10.3 (a) $n\mu$ and $n\sigma^2$, where $\mu := \mu(f_1) - \mu(f_2)$ and $\sigma^2 := \sigma^2(f_1) + \sigma^2(f_2) - 2\mathrm{Cov}(\ln(1+f_1X), \ln(1+f_2X))$. (b) $n\mu$, where μ is as in (a), and $n\sigma_\circ^2$, where $\sigma_\circ^2 := \sigma^2(f_1) + \sigma^2(f_2)$.

10.5 Under mild assumptions, $\lim_{n\to\infty} n^{-1} \ln \mathrm{P}(F_n(f) \leq F_0) = \ln \rho$, where $\rho := \inf_t \mathrm{E}[(1+fX)^{-t}]$.

10.6 (b) Consider the example in which X assumes values -1, 0, and 100 with probabilities 0.5, 0.49, and 0.01, resp.

10.7 See Table B.2.

10.8 Let $a = -a_1 a_2 (p_1 + p_2 + q) < 0$, $b = a_1(a_2 - 1)p_1 + (a_1 - 1)a_2 p_2 - (a_1 + a_2)q$, and $c = a_1 p_1 + a_2 p_2 - q > 0$, and find the positive root of the quadratic $af^2 + bf + c = 0$.

10.9 0.000341966; 0.000294937; yes.

10.12 (c) is best; (a) is worst.

10.13 $f_1^* = [\mu_1(1-\mu_2^2)]/(1-\mu_1^2\mu_2^2)$ and $f_2^* = [\mu_2(1-\mu_1^2)]/(1-\mu_1^2\mu_2^2)$.

10.14 The exact f^* is the root of the equation $\mathrm{E}[S/(1+fS)] = 0$, where S is twice a binomial(d,p) less d. 1: 0.080000000; 0.080000000; 2: 0.079491256; 0.079491256; 3: 0.078988942 (approx.); 0.078976335 (exact); 4: 0.078492936; 0.078454419; 5: 0.078003120; 0.077924498; 6: 0.077519380; 0.077385294; 7: 0.077041602; 0.076835138; 8: 0.076569678; 0.076271759; 9: 0.076103501; 0.075691869; 10: 0.075642965; 0.075090286.

10.15 $f_1^* = (p-q)/(1+(p-q)^2+4c)$.

Table B.2 Extension of Table 10.1 (Problem 10.7).

n	$L_n(f, 0.025)$	$U_n(f, 0.025)$	$L_n(f, 0.05)$
	$f = (1/3)f^*$		
10	.960588	1.04335	.966991
100	.887313	1.15232	.906151
1,000	.739265	1.68932	.790046
10,000	.822372	11.2213	1.01463
100,000	1074.01	$4.16977 \cdot 10^{6}$	2087.02
1,000,000	$3.80817 \cdot 10^{42}$	$8.52048 \cdot 10^{53}$	$3.11243 \cdot 10^{43}$
	$f = (2/3)f^*$		
10	.922315	1.08809	.934652
100	.783824	1.32198	.817461
1,000	.522739	2.72984	.597024
10,000	.433599	80.7493	.660044
100,000	13547.8	$2.04362 \cdot 10^{11}$	51160.7
1,000,000	$7.26263 \cdot 10^{65}$	$3.64416 \cdot 10^{88}$	$4.8522 \cdot 10^{67}$
	$f = f^*$		
10	.885168	1.13426	.902989
100	.689319	1.50992	.734167
1,000	.353537	4.21982	.431521
10,000	.146549	372.658	.275252
100,000	2005.27	$1.17697 \cdot 10^{14}$	14717.6
1,000,000	$6.88737 \cdot 10^{69}$	$7.78631 \cdot 10^{103}$	$3.76291 \cdot 10^{72}$
	$f = (4/3)f^*$		
10	.84913	1.18187	.872002
100	.603497	1.71698	.656415
1,000	.228680	6.24006	.298305
10,000	$3.17413 \cdot 10^{-2}$	1102.89	$7.35629 \cdot 10^{-2}$
100,000	3.47524	$7.95421 \cdot 10^{14}$	49.5819
1,000,000	$3.18688 \cdot 10^{54}$	$8.17426 \cdot 10^{99}$	$1.42464 \cdot 10^{58}$
	$f = (5/3)f^*$		
10	.814184	1.23095	.841693
100	.525991	1.94387	.584270
1,000	.141460	8.82721	.197217
10,000	$4.40387 \cdot 10^{-3}$	2092.80	$1.25947 \cdot 10^{-2}$
100,000	$7.02838 \cdot 10^{-5}$	$6.29167 \cdot 10^{13}$	$1.94971 \cdot 10^{-3}$
1,000,000	$6.97656 \cdot 10^{19}$	$4.09797 \cdot 10^{76}$	$2.55373 \cdot 10^{24}$

10.16 Without loss of generality, assume that $(a_1+1)p_1 \geq (a_2+1)p_2$ and $(a_1+1)p_1 - 1 > 0$. We can make $a_3 > 0$ small enough that the Kelly bettor never bets on outcome 3. If $r_2 := p_1 + [1-(a_1+1)^{-1}](a_2+1)p_2 \leq 1$, then $f_1^* = [(a_1+1)p_1 - 1]/a_1$ and $f_2^* = f_3^* = 0$. If $r_2 > 1$, then $f_1^* = p_1 - (a_1+1)^{-1}w_2$, $f_2^* = p_2 - (a_2+1)^{-1}w_2$, and $f_3^* = 0$, where $w_2 = (1-p_1-p_2)/[1-(a_1+1)^{-1} - (a_2+1)^{-1}]$.

10.17 Take $p_1 = p_2 = p_3 = 1/3$ and $a_1 = 1$, and let $a_2 \in (0,1)$ vary.

10.18 $f_1^* = 2p - 1 + c$ and $f_2^* = c$, where $0 \leq c \leq 1 - p$.

10.19 If $a_1 a_2 < 1$, then $f_1^* = \frac{1}{2}(1 - 1/a_1)$ and $f_2^* = 0$. If $a_1 a_2 > 1$, then $f_1^* = f_2^* = \frac{1}{2}$. If $a_1 a_2 = 1$, then $f_2^* = c$ and $f_1^* = a_2 c + \frac{1}{2}(1 - 1/a_1)$, where $0 \leq c \leq (a_2+1)^{-1}\frac{1}{2}(1 + 1/a_1)$.

10.22 Consider $p_1 = 3/37$, $p_2 = \cdots = p_{35} = 1/37$, $p_{36} = p_{37} = 0$.

10.23 Use Example 10.2.3 with $d = 2$. For example, take $a_1 = 5/3$, $a_2 = 2/3$, $p_1 = 3/7$, and $p_2 = 4/7$.

Chapter 11

11.2 See Table B.3.

11.5 $\frac{1}{2} + (\frac{1}{2})^N$; $(a^{N-1} + (a-1)^{N-1} + (a-2)^{N-1} + \cdots + 1^{N-1})/a^N$.

11.6 $1 - 1/N$.

11.7 $\left\langle {N \atop 1} \right\rangle = 1$ and $\left\langle {N \atop 2} \right\rangle = 2^N - N - 1$, so the distance of interest is $1 - (2^N - N)/N!$. For $N = 52$, this is $1 - 0.558356 \cdot 10^{-52}$.

11.9 (a) k cards must be dealt with probability

$$\left(1 - \frac{m}{N}\right)\left(1 - \frac{m}{N-1}\right)\cdots\left(1 - \frac{m}{N-k+2}\right)\frac{m}{N-k+1}.$$

11.11 $4!\,(13)^4/(52)_4 \approx 0.105498$.

11.12 0.486279.

11.13 No; yes.

11.14 See Table B.4

11.16 Let O_n and E_n be the numbers of odd and even cards among the first n. Then $Z_n = [(O_n - E_n)^2 - (N-n)]/(N-n)_2$ and $\mathrm{E}[Z_n] = -1/(N-1)$.

11.17 (a) $-\frac{1}{2}(4d-1)/(52d-1)$; going to war is superior. (b) $(4d-1)(520d^2 - 14d - 3)/[2(26d-1)(28d-1)(52d-3)]$; there are no pushes. (c) $(4d-1)(208d^2 + 10d - 3)/[(26d-1)(28d-1)(52d-3)]$.

11.18 (a) Yes, when all remaining cards are of the same denomination, the player has a sure win, for example. (b)

Table B.3 The multinomial 3-shuffle of a deck of size $N = 4$, initially in natural order (Problem 11.2).

break	probab.	equally likely card orders after shuffle
||1234	1/81	1234
|1|234	4/81	1234, 2134, 2314, 2341
|12|34	6/81	1234, 1324, 1342, 3124, 3142, 3412
|123|4	4/81	1234, 1243, 1423, 4123
|1234|	1/81	1234
1||234	4/81	1234, 2134, 2314, 2341
1|2|34	12/81	1234, 1324, 1342, 2134, 2314, 2341, 3124, 3142, 3214, 3241, 3412, 3421
1|23|4	12/81	1234, 1243, 1423, 2134, 2143, 2314, 2341, 2413, 2431, 4123, 4213, 4231
1|234|	4/81	1234, 2134, 2314, 2341
12||34	6/81	1234, 1324, 1342, 3124, 3142, 3412
12|3|4	12/81	1234, 1243, 1324, 1342, 1423, 1432, 3124, 3142, 3412, 4123, 4132, 4312
12|34|	6/81	1234, 1324, 1342, 3124, 3142, 3412
123||4	4/81	1234, 1243, 1423, 4123
123|4|	4/81	1234, 1243, 1423, 4123
1234||	1/81	1234

Chapter 12

12.1 0: 0.879168; 3: 0.072960; 5: 0.018240; 10: 0.018048; 14: 0.007168; 18: 0.003456; 150: 0.000768; 300: 0.000128; 1,000: 0.000064; mean: 0.870720; variance: 97.483959.

12.2 0: 0.911; 3: 0.044625; 5: 0.019125; 10: 0.011025; 14: 0.0048; 18: 0.00223125; 20: 0.005675; 150: 0.0015125; 5,000: 0.00000625; mean: 0.818738; variance: 195.526894.

12.3 (a)

$$\mathrm{E}[R] = \frac{1}{n^r} \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} p(s(1, j_1), \ldots, s(r, j_r))$$

and

$$\mathrm{Var}(R) = \frac{1}{n^r} \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} p(s(1, j_1), \ldots, s(r, j_r))^2 - (\mathrm{E}[R])^2.$$

or (b)

Table B.4 Moments in Example 11.3.2 (Problem 11.14).

n	mean	variance	2nd mom.	n	mean	variance	2nd mom.
0	.000000000	.000000000	.000000000	26	.109062752	.007713159	.019607843
1	.019607843	.000000000	.000384468	27	.117787773	.007302511	.021176471
2	.019607843	.000399846	.000784314	28	.117787773	.009001858	.022875817
3	.030012005	.000299760	.001200480	29	.127298587	.008518003	.024722933
4	.030012005	.000733266	.001633987	30	.127298587	.010533038	.026737968
5	.038313198	.000618040	.002085941	31	.137805771	.009954481	.028944911
6	.038313198	.001089644	.002557545	32	.137805771	.012382118	.031372549
7	.045692036	.000962347	.003050109	33	.149591791	.011678023	.034055728
8	.045692036	.001477300	.003565062	34	.149591791	.014659333	.037037037
9	.052598971	.001337315	.004103967	35	.163049876	.013783827	.040369089
10	.052598971	.001901882	.004668534	36	.163049876	.017532385	.044117647
11	.059270060	.001747701	.005260641	37	.178750976	.016414102	.048366013
12	.059270060	.002369413	.005882353	38	.178750976	.021269377	.053221289
13	.065855623	.002198985	.006535948	39	.197566868	.019790862	.058823529
14	.065855623	.002886979	.007223942	40	.197566868	.026326810	.065359477
15	.072466612	.002697716	.007949126	41	.220915679	.024280042	.073083779
16	.072466612	.003463187	.008714597	42	.220915679	.033549204	.082352941
17	.079195654	.003251858	.009523810	43	.251306196	.030527113	.093681917
18	.079195654	.004108671	.010380623	44	.251306196	.044688333	.107843137
19	.086128607	.003871227	.011289364	45	.293734515	.039770455	.126050420
20	.086128607	.004836765	.012254902	46	.293734515	.064046832	.150326797
21	.093352297	.004568081	.013282732	47	.360144058	.054609983	.184313725
22	.093352297	.005664434	.014379085	48	.360144058	.105590375	.235294118
23	.100960948	.005357935	.015551048	49	.490196078	.079969243	.320261438
24	.100960948	.006613610	.016806723	50	.490196078	.249903883	.490196078
25	.109062752	.006260726	.018155410	51	1.000000000	.000000000	1.000000000

$$\mathrm{E}[R] = \frac{1}{n^r}\sum_{k_1=1}^{m}\cdots\sum_{k_r=1}^{m} f(1,k_1)\cdots f(r,k_r)p(k_1,\ldots,k_r)$$

and

$$\mathrm{Var}(R) = \frac{1}{n^r}\sum_{k_1=1}^{m}\cdots\sum_{k_r=1}^{m} f(1,k_1)\cdots f(r,k_r)p(k_1,\ldots,k_r)^2 - (\mathrm{E}[R])^2.$$

12.4 In the following list, the example "bell/orange/bell appears 0 times on reel 1, 3 times on reel 2, and 4 times on reel 3" appears as the entry labeled by an asterisk.
1 3 5: 3 0 0; 1 4 2: 0 3 0; 1 6 3: 0 1 0; 2 3 1: 3 0 0; 2 4 3: 0 0 4; 2 4 5: 0 3 0; 2 6 3: 0 0 1; 2 6 4: 1 0 0; 3 1 3: 3 0 0; 3 2 3: 3 0 0; 3 2 6: 1 0 0; 3 4 1: 0 1 0; 3 4 2: 0 0 4; 3 4 5: 0 0 1; 3 5 3: 3 0 0; 4 1 4: 0 3 0; 4 1 6: 0 1 0; 4 2 4: 0 3 4*; 4 2 6: 0 0 1; 4 3 2: 1 0 0; 4 3 4: 0 0 4; 4 5 4: 0 3 1; 5 3 2: 3 0 0; 5 4 1: 0 3 0; 5 4 2: 0 0 1; 6 3 4: 0 1 1; 6 4 3: 1 0 0.

12.5 cov12, cov13, cov14, cov15, cov23, cov24, cov25, cov34, cov35, cov45 are respectively $-0.518220, -0.518220, -0.518220, -0.724083, -0.342412, 0.230090, -0.446848, -0.240985, 0.024226, -0.382611$; variance: 36.922649.

12.6 0.760803; 23.579504; 0.854950; 36.188693.

12.7 0.865761; 82.116189; 0.865772; 82.143623; 0.874673; 117.850652.

12.8 1.199756; 1.199850; 1.331330.

12.9 Table 12.14. $\mu \approx 0.874673$; $\sigma \approx 10.855904$; for $n = 10^3$, 0.310005 to 1.439341; for $n = 10^4$, 0.696110 to 1.053237; for $n = 10^5$, 0.818206 to 0.931140. Table 12.15. $W = 20$: 0.0922562. $W = 50$: 0.168728. $W = 100$: 0.261449. $W = 200$: 0.293266. $W = 500$: 0.223763.

12.10 1 coin: 0.099992; 63.693103. 2 coins: 0.192679; 35.443737. 3 coins: 0.268939; 26.294141. 4 coins: 0.308162; 22.529822.

12.11 With m being the number of coins played, and assume that there is positive probability of losing the m coins in a single coup. Then $mL/\mathrm{H}_0(X) \le E[N] \le m(L+m-1)/\mathrm{H}_0(X)$.

12.12 (a) The transition probabilities are as in (4.202) with $m = 10$ and $(p_0, p_1, \ldots, p_9) = (p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{O}, p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{E}, p_\mathrm{O})$, where $p_\mathrm{E} = 0.032$ and $p_\mathrm{O} = 0.643$. For the stationary distribution, see Table B.5. Thus, the expected payout, at equilibrium, is about 0.838811. (b) From state (i, j), the expectation is positive unless $(i, j) = (2, 2)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(7, 1)$, $(0, 0)$, $(1, 0)$, $(2, 0)$, $(6, 0)$, or $(7, 0)$.

Table B.5 Stationary distribution of the Markov chain, rounded to six decimal places. Rows indicate cam position, and columns indicate pointer position (Problem 12.12).

	0	1	2	3	4	5	6	7	8	9	sum
0	.071306	.001267	.001226	.001187	.023090	.000410	.000397	.000384	.000372	.000360	1/10
1	.003549	.069024	.001226	.001187	.001149	.022351	.000397	.000384	.000372	.000360	1/10
2	.003549	.003435	.066815	.001187	.001149	.001112	.021636	.000384	.000372	.000360	1/10
3	.003549	.003435	.003325	.064677	.001149	.001112	.001077	.020943	.000372	.000360	1/10
4	.003549	.003435	.003325	.003219	.062608	.001112	.001077	.001042	.020273	.000360	1/10
5	.003549	.003435	.003325	.003219	.003116	.060604	.001077	.001042	.001009	.019624	1/10
6	.071306	.001267	.001226	.001187	.001149	.001112	.021636	.000384	.000372	.000360	1/10
7	.003549	.069024	.001226	.001187	.001149	.001112	.001077	.020943	.000372	.000360	1/10
8	.003549	.003435	.066815	.001187	.001149	.001112	.001077	.001042	.020273	.000360	1/10
9	.003549	.003435	.003325	.064677	.001149	.001112	.001077	.001042	.001009	.019624	1/10
sum	.171001	.161193	.151837	.142915	.096857	.091152	.050526	.047593	.044797	.042130	

12.14 0.611763.

Chapter 13

13.3 37- or 38-number wheels: no. 36-number wheel: yes, there are examples with $(|A|, |B|, |A \cap B|) = (24, 18, 12), (24, 6, 4), (24, 3, 2), (18, 12, 6), (18, 6, 3), (18, 4, 2), (18, 2, 1), (12, 12, 4), (12, 6, 2), (12, 3, 1)$.

13.4 See Table B.6.

Table B.6 Correlations between outside bets. Rows and columns are ordered as follows: low, high, odd, even, red, black; first, second, third dozen; first, second, third column (Problem 13.4).

38-number wheel											
1.000	−.900	.050	.050	.050	.050	.716	.036	−.645	.036	.036	.036
−.900	1.000	.050	.050	.050	.050	−.645	.036	.716	.036	.036	.036
.050	.050	1.000	−.900	.156	−.056	.036	.036	.036	.036	.036	.036
.050	.050	−.900	1.000	−.056	.156	.036	.036	.036	.036	.036	.036
.050	.050	.156	−.056	1.000	−.900	.036	.036	.036	.036	−.191	.263
.050	.050	−.056	.156	−.900	1.000	.036	.036	.036	.036	.263	−.191
.716	−.645	.036	.036	.036	.036	1.000	−.462	−.462	.026	.026	.026
.036	.036	.036	.036	.036	.036	−.462	1.000	−.462	.026	.026	.026
−.645	.716	.036	.036	.036	.036	−.462	−.462	1.000	.026	.026	.026
.036	.036	.036	.036	.036	.036	.026	.026	.026	1.000	−.462	−.462
.036	.036	.036	.036	−.191	.263	.026	.026	.026	−.462	1.000	−.462
.036	.036	.036	.036	.263	−.191	.026	.026	.026	−.462	−.462	1.000
37-number wheel											
1.000	−.947	.026	.026	.026	.026	.712	.019	−.674	.019	.019	.019
−.947	1.000	.026	.026	.026	.026	−.674	.019	.712	.019	.019	.019
.026	.026	1.000	−.947	.135	−.082	.019	.019	.019	.019	.019	.019
.026	.026	−.947	1.000	−.082	.135	.019	.019	.019	.019	.019	.019
.026	.026	.135	−.082	1.000	−.947	.019	.019	.019	.019	−.212	.250
.026	.026	−.082	.135	−.947	1.000	.019	.019	.019	.019	.250	−.212
.712	−.674	.019	.019	.019	.019	1.000	−.480	−.480	.013	.013	.013
.019	.019	.019	.019	.019	.019	−.480	1.000	−.480	.013	.013	.013
−.674	.712	.019	.019	.019	.019	−.480	−.480	1.000	.013	.013	.013
.019	.019	.019	.019	.019	.019	.013	.013	.013	1.000	−.480	−.480
.019	.019	.019	.019	−.212	.250	.013	.013	.013	−.480	1.000	-.480
.019	.019	.019	.019	.250	−.212	.013	.013	.013	−.480	−.480	1.000
36-number wheel											
1.000	−1.000	.000	.000	.000	.000	.707	.000	−.707	.000	.000	.000
−1.000	1.000	.000	.000	.000	.000	−.707	.000	.707	.000	.000	.000
.000	.000	1.000	−1.000	.111	−.111	.000	.000	.000	.000	.000	.000
.000	.000	−1.000	1.000	−.111	.111	.000	.000	.000	.000	.000	.000
.000	.000	.111	−.111	1.000	−1.000	.000	.000	.000	.000	−.236	.236
.000	.000	−.111	.111	−1.000	1.000	.000	.000	.000	.000	.236	- .236
.707	−.707	.000	.000	.000	.000	1.000	−.500	−.500	.000	.000	.000
.000	.000	.000	.000	.000	.000	−.500	1.000	−.500	.000	.000	.000
−.707	.707	.000	.000	.000	.000	−.500	−.500	1.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000	−.500	−.500
.000	.000	.000	.000	−.236	.236	.000	.000	.000	−.500	1.000	−.500
.000	.000	.000	.000	.236	−.236	.000	.000	.000	−.500	−.500	1.000

13.5 With z_1, z_2, z_3 being the (real) roots of the cubic equation $z^3 + z^2 - (q/p)z - 1 = 0$, where $0 < p < \frac{1}{2}$ and $q := 1 - p$, solve the linear system

$$\begin{aligned} c_0 + c_1 z_1^{-2L} + c_2 z_2^{-2L} + c_3 z_3^{-2L} &= 0, \\ c_0 + c_1 z_1^{-2L-1} + c_2 z_2^{-2L-1} + c_3 z_3^{-2L-1} &= 0, \\ c_0 + c_1 z_1^{2W} + c_2 z_2^{2W} + c_3 z_3^{2W} &= 1, \\ c_0 + c_1 z_1^{2W+1} + c_2 z_2^{2W+1} + c_3 z_3^{2W+1} &= 1, \end{aligned}$$

for c_0, c_1, c_2, c_3 using Cramer's rule. Then the probability of reaching the goal is $c_0 + c_1 + c_2 + c_3$.

13.6 $(36 + z)(1 - [1 - 1/(36 + z)]^n)$; 132.

13.7 $\mathrm{P}(X = 37 - j) = \sum_{k=j}^{37} (-1)^{k-j} \binom{k}{j} \binom{37}{k} (1 - k/37)^n$; 24.

13.8 155.458690; 45.386689.

13.9 Win one unit with every number except 13; lose 143 units with number 13.

13.10 Numbers $0, 00, 1, 2, \ldots, 36$ pay -17; -17; -13; -10; -13; -13; -10; -13; -13; -10; -13; -13; -8; -15; 7; 33; 7; 48; 163; 48; 3; 33; 3; -12; -9; -12; -15; -12; -15; -15; -10; -17; -13; -14; -13; -17; -10; -17, respectively. Required ratio is $-1/19$, as expected.

13.11 1/70; 0.014085; 0.986397.

13.12 (a) 0.486486486; 0.492968172; 0.493055696; 0.493056878. (b) (13.10) or 0.493056895.

13.13 Table 13.1: Single number: 0.111111; 2.514157; four numbers: 0.111111; 0.993808. Table 13.2: Single number: 0.367047; 0.0338135; $0.510076 \cdot 10^{-14}$; four numbers: 0.0969629; $0.203704 \cdot 10^{-9}$; $0.123023 \cdot 10^{-96}$.

13.16 See Table B.7.

Chapter 14

14.1 19,958,144,160; 0.

14.2 See Table B.8.

14.3 0.00176410.

14.4 0.220630.

14.5 0.292557; 24.669881.

14.6 0.285967; 18.867072.

14.7 (a) See Table B.9.

(b) $m = 4$: 0.310737; 12.434193. $m = 5$: 0.290583; 27.925895. $m = 6$: 0.290583; 62.460316.

14.8 See Table B.10.

14.10 (a) 170.404571; 1224.821963. (b) See Table B.11. 169.920342; 1,162.307776.

14.11 0.000026128; 0.000028277.

Table B.7 Number of coups needed to conclude that a favorable number exists. Observed proportion of occurrences of the most frequent number is $1/k$. Columns are labeled by α, where $100(1-\alpha)\%$ is the confidence level (Problem 13.16).

k	.01	.02	.05	.10	.20	.50
19	522	465	392	337	282	212
20	652	582	490	421	353	265
21	818	730	614	528	443	333
22	1,031	920	774	665	558	419
23	1,306	1,166	981	843	707	531
24	1,669	1,489	1,253	1,077	903	679
25	2,156	1,923	1,619	1,391	1,166	876
26	2,821	2,517	2,118	1,820	1,526	1,147
27	3,755	3,350	2,820	2,423	2,032	1,526
28	5,111	4,560	3,838	3,298	2,766	2,077
29	7,161	6,389	5,377	4,621	3,875	2,910
30	10,431	9,305	7,832	6,730	5,644	4,239
31	16,038	14,308	12,042	10,348	8,678	6,517
32	26,703	23,821	20,048	17,229	14,447	10,851
33	50,484	45,036	37,903	32,572	27,314	20,514
34	120,577	107,565	90,528	77,796	65,238	48,995
35	511,095	455,941	383,726	329,758	276,525	207,677

14.12 See Table B.12. Assuming a bet of 91,390, expected return is 62922.724 without MAP; 41274.677 with MAP.

14.13 Result is nonrandom.

14.14 71,678.916.

14.15 87,572.130; 87,456.550.

14.16 2.754031; 130.939167; 2.754031; 130.960887.

14.17 (a) See Table B.13. Median is 74; 0.000000625470. (b) See Table B.14. Median is 42.

Chapter 15

15.1 (a) hardways: 4 or 10, 0.111111; 6 or 8, 0.090909. (b) place bets: 4 or 10, 0.066667; 5 or 9, 0.040000; 6 or 8, 0.015152; buy bets, commission always: 4 or 10, 0.047619; 5 or 9, 0.047619; 6 or 8, 0.047619; buy bets, commission on win only: 4 or 10, 0.016667; 5 or 9, 0.020000; 6 or 8, 0.022727. (c) place

Table B.8 Relationship between bet size (rows), maximum aggregate payout (columns), and house advantage (entries) (Problem 14.2).

	50K	100K	250K	500K	1M	2.5M	5M
1	.311492	.311492	.311492	.311492	.311492	.311492	.311492
2	.333221	.311492	.311492	.311492	.311492	.311492	.311492
5	.398405	.354949	.311492	.311492	.311492	.311492	.311492
10	.420134	.398405	.333221	.311492	.311492	.311492	.311492
20	.430998	.420134	.387541	.333221	.311492	.311492	.311492
50	.517744	.433171	.420134	.398405	.354949	.311492	.311492
100	.600144	.517744	.430998	.420134	.398405	.333221	.311492

Table B.9 The probability of catching all m spots (Problem 14.7).

m	probability	reciprocal
1	.250000	4.000
2	$.601266 \cdot 10^{-01}$	16.632
3	$.138754 \cdot 10^{-01}$	72.070
4	$.306339 \cdot 10^{-02}$	326.436
5	$.644925 \cdot 10^{-03}$	1,550.569
6	$.128985 \cdot 10^{-03}$	7,752.843
7	$.244026 \cdot 10^{-04}$	40,979.314
8	$.434566 \cdot 10^{-05}$	230,114.608
9	$.724277 \cdot 10^{-06}$	1,380,687.647
10	$.112212 \cdot 10^{-06}$	8,911,711.176
11	$.160303 \cdot 10^{-07}$	62,381,978.235
12	$.209090 \cdot 10^{-08}$	478,261,833.137
13	$.245989 \cdot 10^{-09}$	4,065,225,581.667
14	$.257003 \cdot 10^{-10}$	38,910,016,281.667
15	$.233639 \cdot 10^{-11}$	428,010,179,098.336

bets to lose: 4 or 10, 0.030303; 5 or 9, 0.025000; 6 or 8, 0.018182; lay bets, commission always: 4 or 10, 0.024390; 5 or 9, 0.032258; 6 or 8, 0.040000.

15.2 $m_j = 4$ if $j = 4, 5, 9, 10$; $m_j = 5$ if $j = 6, 8$.

15.3 $5, 9$: 0.0151675; $6, 8$: 0.0234287; 7: 0.0141414; random: 0.0183532.

15.4 9.023765.

15.5 $\mathrm{P}(X = 1,\, D = 1) = \pi_7 + \pi_{11}$, $\mathrm{P}(X = -1,\, D = 1) = \pi_2 + \pi_3 + \pi_{12}$, and, for $n \geq 2$,

Table B.10 8-spot payoffs obtained from 10-spot ones.

casino	0–3	4	5	6	7	8
Excalibur	0	1.640	13.279	89.836	509.167	2,464.789
Harrah's et al.	0	1.080	13.856	98.129	553.071	3,133.803
Imperial Palace	0	1.524	14.122	91.481	505.820	2,593.897
Sahara	0	1.640	13.279	89.836	509.167	2,464.789
Treasure Island	0	1.383	13.406	90.340	492.399	2,380.751

$$\mathrm{P}(X=1,\,D=n)=\sum_{j\in\mathscr{P}}\pi_j(1-\pi_j-\pi_7)^{n-2}\pi_j,$$
$$\mathrm{P}(X=-1,\,D=n)=\sum_{j\in\mathscr{P}}\pi_j(1-\pi_j-\pi_7)^{n-2}\pi_7.$$

$\mathrm{E}[X \mid D=1]=1/3$ and, for $n\ge 2$,

$$\mathrm{E}[X \mid D=n]=\frac{\sum_{j\in\mathscr{P}}\pi_j(1-\pi_j-\pi_7)^{n-2}(\pi_j-\pi_7)}{\sum_{j\in\mathscr{P}}\pi_j(1-\pi_j-\pi_7)^{n-2}(\pi_j+\pi_7)},$$

hence $\lim_{n\to\infty}\mathrm{E}[X \mid D=n]=-1/3$. Finally, $\mathrm{E}[D \mid X=1]=2.938301$ and $\mathrm{E}[D \mid X=-1]=3.801014$.

15.6 For the pass line, $\mathrm{H}_0=14/(990+165m_4+220m_5+275m_6)$. For don't pass, $\mathrm{H}_0=9/[660+220(m_4+m_5+m_6)]$ and $\mathrm{H}=27/[1{,}925+660(m_4+m_5+m_6)]$.

15.7 -0.00614789.

15.8 0.957775.

15.9 7/9; $n=2$.

15.10 $E\approx 8.525510$; $E_4\approx 6.841837$; $E_5\approx 7.010204$; $E_6\approx 7.147959$.

15.12 $s(1)=0$ and, for $n\ge 2$,

$$s(n)=\Big(1-\sum_{j\in\mathscr{P}}\pi_j\Big)s(n-1)+\sum_{j\in\mathscr{P}}\pi_j(1-\pi_j-\pi_7)^{n-2}\pi_7 + \sum_{j\in\mathscr{P}}\pi_j\sum_{l=2}^{n-1}(1-\pi_j-\pi_7)^{l-2}\pi_j\, s(n-l).$$

15.13 geometric$(q/(q+\pi_7))$, where q is as in (15.18). Mean is 1.420918, variance is 0.598091.

15.14 60.763636; 121.527272.

15.15 (a) $4,10$: 0.070153; $5,9$: 0.112245; $6,8$: 0.159439. (b) shifted geometric$(q/(1-(1-p)(1-q)))$, where $p=\pi_j^A$ as in (15.30) and q is as in (15.18). (c)

Table B.11 252-way 10-spot ticket (Problem 14.10).

n_0	n_1	n_2	pay	probability	n_0	n_1	n_2	pay	probability
10	0	0	0	$.118571 \cdot 10^{-02}$	4	3	3	6,216	$.325706 \cdot 10^{-02}$
9	1	0	0	$.115679 \cdot 10^{-01}$	3	4	3	10,980	$.143310 \cdot 10^{-02}$
8	2	0	0	$.470977 \cdot 10^{-01}$	2	5	3	17,748	$.337201 \cdot 10^{-03}$
7	3	0	0	$.105148 \cdot 10^{+00}$	1	6	3	26,970	$.389078 \cdot 10^{-04}$
6	4	0	0	$.142189 \cdot 10^{+00}$	0	7	3	39,165	$.167797 \cdot 10^{-05}$
5	5	0	0	$.121335 \cdot 10^{+00}$	6	0	4	7,380	$.831228 \cdot 10^{-04}$
4	6	0	0	$.659429 \cdot 10^{-01}$	5	1	4	13,920	$.244279 \cdot 10^{-03}$
3	7	0	0	$.224486 \cdot 10^{-01}$	4	2	4	23,904	$.268707 \cdot 10^{-03}$
2	8	0	0	$.455988 \cdot 10^{-02}$	3	3	4	37,818	$.140500 \cdot 10^{-03}$
1	9	0	0	$.496313 \cdot 10^{-03}$	2	4	4	56,240	$.364761 \cdot 10^{-04}$
0	10	0	0	$.218378 \cdot 10^{-04}$	1	5	4	79,840	$.440466 \cdot 10^{-05}$
9	0	1	0	$.261654 \cdot 10^{-02}$	0	6	4	109,380	$.190325 \cdot 10^{-06}$
8	1	1	0	$.197153 \cdot 10^{-01}$	5	0	5	67,300	$.537414 \cdot 10^{-05}$
7	2	1	0	$.609383 \cdot 10^{-01}$	4	1	5	92,380	$.105375 \cdot 10^{-04}$
6	3	1	0	$.101112 \cdot 10^{+00}$	3	2	5	125,380	$.729522 \cdot 10^{-05}$
5	4	1	23	$.989143 \cdot 10^{-01}$	2	3	5	167,110	$.220233 \cdot 10^{-05}$
4	5	1	115	$.589277 \cdot 10^{-01}$	1	4	5	218,495	$.285487 \cdot 10^{-06}$
3	6	1	345	$.212794 \cdot 10^{-01}$	0	5	5	280,575	$.124576 \cdot 10^{-07}$
2	7	1	805	$.446682 \cdot 10^{-02}$	4	0	6	302,760	$.151984 \cdot 10^{-06}$
1	8	1	1,610	$.491350 \cdot 10^{-03}$	3	1	6	370,380	$.183527 \cdot 10^{-06}$
0	9	1	2,898	$.214096 \cdot 10^{-04}$	2	2	6	453,150	$.713718 \cdot 10^{-07}$
8	0	2	0	$.190432 \cdot 10^{-02}$	1	3	6	552,285	$.103814 \cdot 10^{-07}$
7	1	2	0	$.108335 \cdot 10^{-01}$	0	4	6	669,138	$.463453 \cdot 10^{-09}$
6	2	2	138	$.247286 \cdot 10^{-01}$	3	0	7	947,415	$.169933 \cdot 10^{-08}$
5	3	2	495	$.294638 \cdot 10^{-01}$	2	1	7	1,096,305	$.111229 \cdot 10^{-08}$
4	4	2	1,198	$.199495 \cdot 10^{-01}$	1	2	7	1,270,983	$.198623 \cdot 10^{-09}$
3	5	2	2,420	$.781694 \cdot 10^{-02}$	0	3	7	1,473,150	$.929229 \cdot 10^{-11}$
2	6	2	4,380	$.171973 \cdot 10^{-02}$	2	0	8	2,381,288	$.620696 \cdot 10^{-11}$
1	7	2	7,343	$.192686 \cdot 10^{-03}$	1	1	8	2,668,400	$.174231 \cdot 10^{-11}$
0	8	2	11,620	$.833739 \cdot 10^{-05}$	0	2	8	2,996,000	$.901192 \cdot 10^{-13}$
7	0	3	483	$.588776 \cdot 10^{-03}$	1	0	9	5,166,000	$.500662 \cdot 10^{-14}$
6	1	3	1,245	$.245532 \cdot 10^{-02}$	0	1	9	5,670,000	$.339432 \cdot 10^{-15}$
5	2	3	3,075	$.398989 \cdot 10^{-02}$	0	0	10	10,080,000	$.282860 \cdot 10^{-18}$

$$\binom{k_4+k_5+k_6+k_8+k_9+k_{10}}{k_4,k_5,k_6,k_8,k_9,k_{10}}\rho_4^{k_4}\rho_5^{k_5}\rho_6^{k_6}\rho_8^{k_8}\rho_9^{k_9}\rho_{10}^{k_{10}}\bigg(1-\sum_{i\in\mathscr{P}}\rho_i\bigg),$$

where $\rho_i = [\pi_i^2/(\pi_i+\pi_7)]/[q+\sum_{j\in\mathscr{P}}\pi_j^2/(\pi_j+\pi_7)]$.

Table B.12 91,390-way 8-spot ticket (Problem 14.12).

n_0	n_1	n_2	without MAP	with MAP	probability
20	20	0	0	0	$.408853 \cdot 10^{-01}$
21	18	1	5,712	5,712	$.184957 \cdot 10^{+00}$
22	16	2	19,904	19,904	$.321574 \cdot 10^{+00}$
23	14	3	59,086	59,086	$.279629 \cdot 10^{+00}$
24	12	4	159,616	100,000	$.132533 \cdot 10^{+00}$
25	10	5	377,700	100,000	$.349886 \cdot 10^{-01}$
26	8	6	789,392	100,000	$.504644 \cdot 10^{-02}$
27	6	7	1,490,594	100,000	$.373810 \cdot 10^{-03}$
28	4	8	2,597,056	100,000	$.125160 \cdot 10^{-04}$
29	2	9	4,244,376	100,000	$.143862 \cdot 10^{-06}$
30	0	10	6,588,000	100,000	$.239769 \cdot 10^{-09}$

15.16 $W_7, W_{11}, L_2, L_3, L_{12}$: 0.420918; 0.140306; 0.070153; 0.140306; 0.070153. $E_4, E_5, E_6, E_8, E_9, E_{10}$: 0.210459; 0.280612; 0.350765; 0.350765; 0.280612; 0.210459. $W_4, W_5, W_6, W_8, W_9, W_{10}$: 0.070153; 0.112245; 0.159439; 0.159439; 0.112245; 0.070153. L_7: 1.000000. $I_4, I_5, I_6, I_8, I_9, I_{10}$: 0.631378; 0.729592; 0.797194; 0.797194; 0.729592; 0.631378.

15.18 (a) 0.513627. (b) 0.434930.

15.19 0.053824; 4.161472; 7.897850.

15.20 0.014141; 0.013636; 4.042424; 7.972112.

15.22

Chapter 16

16.1 0.0287146; 0.255333.

16.2 (a) 1: 0.837551; 2: 0.092814; 3: 0.069635. (b) 1: 0.845651; 2: 0.084990; 3: 0.069358. (c) 1: 0.834537; 2: 0.094136; 3: 0.071327.

16.3 0.043845 (vs. 0.028453).

16.4 correlations: 0.772056; 0.919440; 0.849856. variance: 26.711089.

16.5 The possible values are 3,000; 600; 400; 150; 100; 33; 24; 22; 16; 15; 10; 9; 6; 5; 4; 3; 2; 1; −1; −2; −3. Associated probabilities are 0.00000153908; 0.00000677194; 0.00000707975; 0.000151445; 0.0000886508; 0.000642719; 0.000192539; 0.000797858; 0.00177286; 0.0000858805; 0.00170561; 0.00701357; 0.0173742; 0.00213316; 0.0131356; 0.0572604; 0.0742931; 0.0621172; 0.742156; 0.0180423; 0.00102110. Variance: 26.711089.

16.6 0.256109; 0.023167.

Table B.13 Distribution of number of numbers needed to coverall (Problem 14.17(a)).

n	distribution	cumulative	n	distribution	cumulative
25	$.931001 \cdot 10^{-18}$	$.969793 \cdot 10^{-18}$	50	$.226324 \cdot 10^{-05}$	$.471508 \cdot 10^{-05}$
26	$.116375 \cdot 10^{-16}$	$.126073 \cdot 10^{-16}$	51	$.419118 \cdot 10^{-05}$	$.890626 \cdot 10^{-05}$
27	$.100858 \cdot 10^{-15}$	$.113466 \cdot 10^{-15}$	52	$.763394 \cdot 10^{-05}$	$.165402 \cdot 10^{-04}$
28	$.680795 \cdot 10^{-15}$	$.794260 \cdot 10^{-15}$	53	$.136884 \cdot 10^{-04}$	$.302287 \cdot 10^{-04}$
29	$.381245 \cdot 10^{-14}$	$.460671 \cdot 10^{-14}$	54	$.241829 \cdot 10^{-04}$	$.544116 \cdot 10^{-04}$
30	$.184268 \cdot 10^{-13}$	$.230336 \cdot 10^{-13}$	55	$.421251 \cdot 10^{-04}$	$.965367 \cdot 10^{-04}$
31	$.789722 \cdot 10^{-13}$	$.102006 \cdot 10^{-12}$	56	$.724025 \cdot 10^{-04}$	$.168939 \cdot 10^{-03}$
32	$.306017 \cdot 10^{-12}$	$.408023 \cdot 10^{-12}$	57	$.122865 \cdot 10^{-03}$	$.291804 \cdot 10^{-03}$
33	$.108806 \cdot 10^{-11}$	$.149608 \cdot 10^{-11}$	58	$.205979 \cdot 10^{-03}$	$.497783 \cdot 10^{-03}$
34	$.359060 \cdot 10^{-11}$	$.508669 \cdot 10^{-11}$	59	$.341337 \cdot 10^{-03}$	$.839120 \cdot 10^{-03}$
35	$.110982 \cdot 10^{-10}$	$.161849 \cdot 10^{-10}$	60	$.559414 \cdot 10^{-03}$	$.139853 \cdot 10^{-02}$
36	$.323698 \cdot 10^{-10}$	$.485547 \cdot 10^{-10}$	61	$.907157 \cdot 10^{-03}$	$.230569 \cdot 10^{-02}$
37	$.896395 \cdot 10^{-10}$	$.138194 \cdot 10^{-09}$	62	$.145623 \cdot 10^{-02}$	$.376192 \cdot 10^{-02}$
38	$.236904 \cdot 10^{-09}$	$.375099 \cdot 10^{-09}$	63	$.231503 \cdot 10^{-02}$	$.607694 \cdot 10^{-02}$
39	$.600158 \cdot 10^{-09}$	$.975256 \cdot 10^{-09}$	64	$.364617 \cdot 10^{-02}$	$.972311 \cdot 10^{-02}$
40	$.146288 \cdot 10^{-08}$	$.243814 \cdot 10^{-08}$	65	$.569158 \cdot 10^{-02}$	$.154147 \cdot 10^{-01}$
41	$.344208 \cdot 10^{-08}$	$.588022 \cdot 10^{-08}$	66	$.880839 \cdot 10^{-02}$	$.242231 \cdot 10^{-01}$
42	$.784030 \cdot 10^{-08}$	$.137205 \cdot 10^{-07}$	67	$.135199 \cdot 10^{-01}$	$.377429 \cdot 10^{-01}$
43	$.173312 \cdot 10^{-07}$	$.310517 \cdot 10^{-07}$	68	$.205871 \cdot 10^{-01}$	$.583300 \cdot 10^{-01}$
44	$.372620 \cdot 10^{-07}$	$.683137 \cdot 10^{-07}$	69	$.311093 \cdot 10^{-01}$	$.894393 \cdot 10^{-01}$
45	$.780728 \cdot 10^{-07}$	$.146387 \cdot 10^{-06}$	70	$.466640 \cdot 10^{-01}$	$.136103 \cdot 10^{+00}$
46	$.159694 \cdot 10^{-06}$	$.306081 \cdot 10^{-06}$	71	$.694996 \cdot 10^{-01}$	$.205603 \cdot 10^{+00}$
47	$.319389 \cdot 10^{-06}$	$.625470 \cdot 10^{-06}$	72	$.102801 \cdot 10^{+00}$	$.308404 \cdot 10^{+00}$
48	$.625470 \cdot 10^{-06}$	$.125094 \cdot 10^{-05}$	73	$.151055 \cdot 10^{+00}$	$.459459 \cdot 10^{+00}$
49	$.120090 \cdot 10^{-05}$	$.245184 \cdot 10^{-05}$	74	$.220541 \cdot 10^{+00}$	$.680000 \cdot 10^{+00}$
50	$.226324 \cdot 10^{-05}$	$.471508 \cdot 10^{-05}$	75	$.320000 \cdot 10^{+00}$	$.100000 \cdot 10^{+01}$

16.7 0.020338 (vs. 0.020147).

16.8 (a) 0.695928.

16.9 A-4-2 with the 4 and the 2 suit-matched. Minimal positive expectation: 0.088417.

16.10 3-2-A: 1.681068; 4-3-2: 1.706470; 5-4-3: 1.705873; 6-5-4: 1.705927; 7-6-5: 1.706795; 8-7-6: 1.707664; 9-8-7: 1.708532; T-9-8: 1.710215; J-T-9: 1.712549; Q-J-T: 1.688667; K-Q-J: 1.662777; A-K-Q: 1.634933; 2-2-2: 1.701205; 3-3-3: 1.700282; 4-4-4: 1.698708; 5-5-5: 1.699577; 6-6-6: 1.700445; 7-7-7: 1.701314; 8-8-8: 1.702182; 9-9-9: 1.703050; T-T-T: 1.706361; J-J-J: 1.709672; Q-Q-Q: 1.625054; K-K-K: 1.625271; A-A-A: 1.626140.

16.11 0.000612068; 0.020161.

Table B.14 Distribution of number of numbers needed to cover any row, any column, or either main diagonal (Problem 14.17(b)).

n	distribution	cumulative	n	distribution	cumulative
4	$.329096 \cdot 10^{-06}$	$.329096 \cdot 10^{-06}$	38	$.349406 \cdot 10^{-02}$	$.371789 \cdot 10^{-01}$
5	$.136274 \cdot 10^{-05}$	$.169183 \cdot 10^{-05}$	39	$.363398 \cdot 10^{-02}$	$.408129 \cdot 10^{-01}$
6	$.352272 \cdot 10^{-05}$	$.521455 \cdot 10^{-05}$	40	$.375019 \cdot 10^{-02}$	$.445631 \cdot 10^{-01}$
7	$.727720 \cdot 10^{-05}$	$.124918 \cdot 10^{-04}$	41	$.383922 \cdot 10^{-02}$	$.484023 \cdot 10^{-01}$
8	$.131405 \cdot 10^{-04}$	$.256322 \cdot 10^{-04}$	42	$.389794 \cdot 10^{-02}$	$.523003 \cdot 10^{-01}$
9	$.216726 \cdot 10^{-04}$	$.473048 \cdot 10^{-04}$	43	$.392375 \cdot 10^{-02}$	$.562240 \cdot 10^{-01}$
10	$.334778 \cdot 10^{-04}$	$.807826 \cdot 10^{-04}$	44	$.391466 \cdot 10^{-02}$	$.601387 \cdot 10^{-01}$
11	$.492032 \cdot 10^{-04}$	$.129986 \cdot 10^{-03}$	45	$.386944 \cdot 10^{-02}$	$.640081 \cdot 10^{-01}$
12	$.695350 \cdot 10^{-04}$	$.199521 \cdot 10^{-03}$	46	$.378770 \cdot 10^{-02}$	$.677958 \cdot 10^{-01}$
13	$.951947 \cdot 10^{-04}$	$.294715 \cdot 10^{-03}$	47	$.366999 \cdot 10^{-02}$	$.714658 \cdot 10^{-01}$
14	$.126932 \cdot 10^{-03}$	$.421648 \cdot 10^{-03}$	48	$.351788 \cdot 10^{-02}$	$.749837 \cdot 10^{-01}$
15	$.165520 \cdot 10^{-03}$	$.587167 \cdot 10^{-03}$	49	$.333390 \cdot 10^{-02}$	$.783176 \cdot 10^{-01}$
16	$.211738 \cdot 10^{-03}$	$.798905 \cdot 10^{-03}$	50	$.312160 \cdot 10^{-02}$	$.814392 \cdot 10^{-01}$
17	$.266367 \cdot 10^{-03}$	$.106527 \cdot 10^{-02}$	51	$.288542 \cdot 10^{-02}$	$.843246 \cdot 10^{-01}$
18	$.330168 \cdot 10^{-03}$	$.139544 \cdot 10^{-02}$	52	$.263059 \cdot 10^{-02}$	$.869552 \cdot 10^{-01}$
19	$.403869 \cdot 10^{-03}$	$.179931 \cdot 10^{-02}$	53	$.236296 \cdot 10^{-02}$	$.893182 \cdot 10^{-01}$
20	$.488136 \cdot 10^{-03}$	$.228745 \cdot 10^{-02}$	54	$.208880 \cdot 10^{-02}$	$.914070 \cdot 10^{-01}$
21	$.583558 \cdot 10^{-03}$	$.287100 \cdot 10^{-02}$	55	$.181455 \cdot 10^{-02}$	$.932215 \cdot 10^{-01}$
22	$.690611 \cdot 10^{-03}$	$.356161 \cdot 10^{-02}$	56	$.154655 \cdot 10^{-02}$	$.947681 \cdot 10^{-01}$
23	$.809634 \cdot 10^{-03}$	$.437125 \cdot 10^{-02}$	57	$.129077 \cdot 10^{-02}$	$.960588 \cdot 10^{-01}$
24	$.940796 \cdot 10^{-03}$	$.531204 \cdot 10^{-02}$	58	$.105251 \cdot 10^{-02}$	$.971114 \cdot 10^{-01}$
25	$.108406 \cdot 10^{-02}$	$.639611 \cdot 10^{-02}$	59	$.836186 \cdot 10^{-03}$	$.979475 \cdot 10^{-01}$
26	$.123916 \cdot 10^{-02}$	$.763526 \cdot 10^{-02}$	60	$.645101 \cdot 10^{-03}$	$.985926 \cdot 10^{-01}$
27	$.140554 \cdot 10^{-02}$	$.904080 \cdot 10^{-02}$	61	$.481289 \cdot 10^{-03}$	$.990739 \cdot 10^{-01}$
28	$.158236 \cdot 10^{-02}$	$.106232 \cdot 10^{-01}$	62	$.345451 \cdot 10^{-03}$	$.994194 \cdot 10^{-01}$
29	$.176846 \cdot 10^{-02}$	$.123916 \cdot 10^{-01}$	63	$.236968 \cdot 10^{-03}$	$.996563 \cdot 10^{-01}$
30	$.196232 \cdot 10^{-02}$	$.143539 \cdot 10^{-01}$	64	$.154008 \cdot 10^{-03}$	$.998104 \cdot 10^{-01}$
31	$.216205 \cdot 10^{-02}$	$.165160 \cdot 10^{-01}$	65	$.937260 \cdot 10^{-04}$	$.999041 \cdot 10^{-01}$
32	$.236540 \cdot 10^{-02}$	$.188814 \cdot 10^{-01}$	66	$.525460 \cdot 10^{-04}$	$.999566 \cdot 10^{-01}$
33	$.256976 \cdot 10^{-02}$	$.214512 \cdot 10^{-01}$	67	$.264961 \cdot 10^{-04}$	$.999831 \cdot 10^{-01}$
34	$.277219 \cdot 10^{-02}$	$.242233 \cdot 10^{-01}$	68	$.115769 \cdot 10^{-04}$	$.999947 \cdot 10^{-01}$
35	$.296943 \cdot 10^{-02}$	$.271928 \cdot 10^{-01}$	69	$.411308 \cdot 10^{-05}$	$.999988 \cdot 10^{-01}$
36	$.315798 \cdot 10^{-02}$	$.303508 \cdot 10^{-01}$	70	$.104887 \cdot 10^{-05}$	$.999999 \cdot 10^{-01}$
37	$.333412 \cdot 10^{-02}$	$.336849 \cdot 10^{-01}$	71	$.139055 \cdot 10^{-06}$	$.100000 \cdot 10^{+00}$

16.12 The possible values are 7; 6; 5; 3; 2; 1; 0; −1; −2. Associated probabilities are 0.00151544; 0.00228890; 0.000710032; 0.0220459; 0.223741; 0.198825; 0.000612068; 0.326456; 0.223805. Variance: 2.687150.

16.13 (a) call. (b) (c)
16.14
16.15

Chapter 17

17.1 2,275 or larger.
17.2 1,080 or larger.
17.3
1. hold A-Q-J-T: 18.361702; hold all: 6.000000.
2. hold all: 4.000000; hold the diamonds: 3.574468; hold Q-J-T: 1.346901.
3. hold the clubs: 2.446809; hold Q-Q: 1.536540.
4. hold all: 50.000000; hold K-Q-J-T: 18.617021.
5. hold A-Q-J: 1.386679; hold the clubs: 1.340426.
6. hold the clubs: 2.468085; hold K-J-T: 1.316374.
7. hold 6-6: 0.823682; hold 9-8-7-6: 0.680851.
8. hold the clubs: 1.276596; hold Q-J-8: 0.590194; hold Q-J: 0.573605.
9. hold A-K-Q-J: 0.595745; hold K-J: 0.582115.
10. hold K-J: 0.591736; hold K-Q-J: 0.515264.
11. hold A-K-Q-J: 0.595745; hold Q-J: 0.593463; hold A-K: 0.567808.
12. hold the hearts: 0.637373; hold A-K-Q-J: 0.595745; hold Q-J: 0.577305; hold A-K: 0.567808.
13. hold the diamonds: 0.525439; hold A-Q: 0.474314; hold the A: 0.463852; hold Q-T: 0.456491; hold the Q: 0.449522.
14. hold A-Q-J-T: 0.531915; hold the spades: 0.522664.
15. hold the hearts: 1.276596; hold A-Q-T: 1.269195.
16. hold the K: 0.459765; hold K-T: 0.458218; hold the hearts: 0.446809.
17. hold J-T-9-8: 0.744681; hold K-J: 0.483195; hold K-J-T-9: 0.468085; hold the J: 0.465826.
18. hold K-K: 1.536540; hold A-K-K, K-K-T, or K-K-5: 1.416281.
19. hold K-J: 0.483195; hold J-T: 0.473265; hold K-J-T-9: 0.468085; hold the K: 0.454759; hold the J: 0.452751.
20. hold the J: 0.484501; hold the hearts: 0.440333.

17.4
1. hold K-Q-J-T: 19.659574; hold all: 9.000000.
2. hold the clubs: 1.382979; hold K-Q-J: 1.301573; hold 9-9: 0.560222.
3. hold 3-3: 0.560222; hold the diamonds: 0.510638.
4. hold the deuces: 15.051804; hold the deuces and one other: 10.382979; hold all: 9.000000.
5. hold T-9-2-2: 3.319149; hold the deuces: 3.255504; hold T-2-2: 2.902868.
6. hold A-K-T: 1.266420; hold K-K: 0.560222; hold the clubs: 0.510638.
7. hold the deuces: 3.265618; hold 6-5-2-2: 3.106383.

8. hold the hearts: 1.283071; hold A-A or K-K: 0.561332; hold A-A-K-K: 0.510638.

9. hold all: 2.000000; hold the diamonds: 1.617021; hold Q-J-T: 1.366327.

10. hold A-J-T-2: 3.361702; hold J-T-9-2: 2.212766; hold all: 2.000000.

11. hold the clubs: 0.355227; hold J-T: 0.352081; hold A-K-J-T: 0.340426; hold nothing: 0.323393.

12. hold Q-J-T-8 or J-T-8-7 or J-8-7: 0.340426; hold Q-T: 0.338514; hold nothing: 0.318520.

13. hold the deuce: 1.048098; hold A-K-2: 1.026827.

14. hold the diamonds: 0.355227; hold Q-T: 0.342461; hold Q-T-9-8: 0.340426; hold nothing: 0.320113.

15. hold K-Q: 0.327845; hold nothing: 0.321177.

16. hold 7-6-2: 1.112858; hold A-J-2: 1.046253; hold the deuce: 1.034357.

17. hold A-K-Q-T: 0.340426; hold Q-T: 0.333210; hold nothing: 0.322912.

18. hold J-T: 0.345174; hold Q-J-T-8 or J-T-8-7 or J-T-7: 0.340426; hold nothing: 0.317861.

19. hold Q-J-T: 1.383904; hold the spades: 1.382979.

20. hold 6-5-4-3 or the hearts: 0.340426; hold nothing: 0.332818.

17.5 (a) For 3, 4, . . . , 9, T, J, Q, K, A, drawing expectations are 15.079556; 15.072155; 15.064755; 15.064755; 15.057354; 15.057354; 15.057354; 14.938945; 14.946346; 14.953747; 14.961147; 14.946346. (b) For 3, 4, . . . , 9, variances are 1,539.174; 1,539.294; 1,539.413; 1,539.413; 1,539.532; 1,539.532; 1,539.532.

17.6 25.837916319.

17.7 (a) See Table B.15. 6; 0.032459; 0.155434; 0.635423; 0.056764; 0.112420; 0.007498. (b) See Table B.16. 6; 0.190664; 0.139791; 0.369840;

Table B.15 Joint distribution of payout and the number of cards held at Jacks or Better (Problem 17.7(a)).

pay	0	1	2	3	4	5
0	.024922245	.103428123	.341506714	.026847719	.048729868	.000000000
1	.005114105	.039612199	.159600055	.005529350	.004729322	.000000000
2	.001519377	.007733163	.075648276	.000884945	.043493142	.000000000
3	.000666603	.003574649	.050973086	.019234361	.000000000	.000000000
4	.000120999	.000473483	.001370600	.000751878	.004615463	.003896943
6	.000063126	.000314522	.000704762	.001279507	.006747216	.001905378
9	.000044945	.000250975	.004439849	.001289989	.004045874	.001440576
25	.000007279	.000045315	.001170772	.000899083	.000000000	.000240096
50	.000000388	.000001211	.000004429	.000037626	.000051805	.000013852
800	.000000072	.000000843	.000004856	.000009785	.000007663	.000001539

0.173075; 0.101858; 0.024771.

Table B.16 Joint distribution of payout and the number of cards held at Deuces Wild (Problem 17.7(b)).

pay	0	1	2	3	4	5
0	.152535310	.063681155	.228082418	.037347175	.065154049	.000000000
1	.027946373	.052320818	.106574722	.094474707	.003227740	.000000000
2	.006247322	.012883843	.006621443	.008225352	.020254976	.018912180
3	.000944906	.001283762	.007586623	.006538051	.000000000	.004875796
5	.002700677	.008562734	.019348658	.023891546	.010434550	.000000000
9	.000184331	.000630720	.000654979	.000843757	.001202772	.000603318
15	.000056431	.000231986	.000677131	.001144228	.000916373	.000175455
25	.000042998	.000161798	.000216940	.000529252	.000660166	.000184689
200	.000005345	.000034484	.000076114	.000069291	.000000000	.000018469
800	.000000266	.000000000	.000001215	.000012024	.000007040	.000001539

17.8 (a) See Table B.17. 18. (b) 0.544430; 1.533884; 3.986924; 15.366482; 200.

Table B.17 Deuces Wild conditional probabilities of the 10 payouts, given the number of deuces (Problem 17.8).

pay	0	1	2	3	4
0	.704672287	.275591572	.000000000	.000000000	0
1	.203195762	.452027604	.383174990	.000000000	0
2	.045718012	.119291920	.182805376	.000000000	0
3	.016168623	.035317199	.000000000	.000000000	0
5	.027865750	.103641386	.358333312	.710217096	0
9	.000967488	.006676809	.032858747	.098431319	0
15	.000944257	.004746654	.025352977	.083895920	0
25	.000426189	.002591707	.015568407	.067543186	0
200	.000008112	.000115150	.001906191	.039912479	1
800	.000033519	.000000000	.000000000	.000000000	0

17.9 (a) 0.0000433262 (vs. 0.0000220839). (b) same.
17.11 The latter is more likely.
17.12

17.13 Solved by John Jungtae Kim, `http://digitalcommons.mcmaster.ca/cgi/viewcontent.cgi?article=7829&context=opendissertations`.

17.14 (b) 150,891. (c) Solved by John Jungtae Kim, op. cit.

Chapter 18

18.1 (a) $(\sum_{i=1}^{13} l_i^2 - m)/[2m(m-1)]$, where $m := l_1 + \cdots + l_{13}$. (b) 0.029412. No.

18.2 (a) $\frac{1}{2}(\sum_{i=1}^{13} l_i^2 - m)/[2eo + \sum_{i=1}^{13} l_i^2 - m]$, where e and o are the numbers of even and odd cards remaining. (b) 0.052156; 0.051867.

18.3 0.025 in the case of (18.4); $0.025(1 - 1/m)$ in the case of (18.11), where m is the number of cards remaining.

18.5 (b) $0.0\overline{95}$; 0.157143.

18.7 $3(m+1)(m-3)/[4m(m-1)(m-2)]$ instead of $3(m-1)/[4m(m-2)]$; 0.014982 instead of 0.015006.

18.8 (b) 1: $-0.117647 \cdot 10^{-02}$; 2: $-0.112845 \cdot 10^{-02}$; 3: $-0.921405 \cdot 10^{-03}$; 4: $-0.680716 \cdot 10^{-03}$; 5: $-0.519749 \cdot 10^{-03}$; 6: $-0.435242 \cdot 10^{-03}$; 7: $-0.364218 \cdot 10^{-03}$; 8: $-0.280870 \cdot 10^{-03}$; 9: $-0.195263 \cdot 10^{-03}$; 10: $-0.122349 \cdot 10^{-03}$; 11: $-0.706271 \cdot 10^{-04}$; 12: $-0.390821 \cdot 10^{-04}$; 13: $-0.216161 \cdot 10^{-04}$; 14: $-0.123787 \cdot 10^{-04}$; 15: $-0.755461 \cdot 10^{-05}$; 16: $-0.503990 \cdot 10^{-05}$; 17: $-0.376106 \cdot 10^{-05}$; 18: $-0.320726 \cdot 10^{-05}$; 19: $-0.319042 \cdot 10^{-05}$; 20: $-0.376424 \cdot 10^{-05}$; 21: $-0.537340 \cdot 10^{-05}$; 22: $-0.969120 \cdot 10^{-05}$; 23: $-0.223216 \cdot 10^{-04}$; 24: $-0.720288 \cdot 10^{-04}$; 25: $-0.117647 \cdot 10^{-02}$.

18.9 $[(k)_2 + (l)_2]/\{2[(m)_2 - (m-k-l)_2 - 2kl]\}$; 0.016854; 0.015464.

18.10 $[(k)_2 + (l)_2]/\{4[(m)_2 - (m-k-l)_2]\}$; 0.008065; 0.015464.

18.11

$$\text{(a)} \quad -\frac{2kl}{(m)_2} - \frac{(k)_2}{2(m)_2}\,\frac{(l)_2}{2(m)_2}; \quad \text{(b)} \quad \frac{2kl}{(m)_2} - \frac{(k)_2}{2(m)_2}\,\frac{(l)_2}{2(m)_2};$$

no; no.

18.12 $26!\,4!^{13}/[2^{13}\,52!]$; 0.534967×10^{-27}.

18.14 1: 0.000000; 2: 0.002353; 3: 0.022569; 4: 0.075630; 5: 0.173445; 6: 0.324850; 7: 0.535606; 8: 0.808403; 9: 1.142857; 10: 1.535510; 11: 1.979832; 12: 2.466218; 13: 2.981993; 14: 3.511405; 15: 4.035630; 16: 4.532773; 17: 4.977863; 18: 5.342857; 19: 5.596639; 20: 5.705018; 21: 5.630732; 22: 5.333445; 23: 4.769748; 24: 3.893157; 25: 2.654118.

Chapter 19

19.2 (a) means: 5.120654; 5.108655. variances: 8.574919; 8.164241. (b) -0.157181.

19.3 (a) See Table B.18.

Table B.18 Conditional expectation of player bet given player's first two cards (Problem 19.3).

player's cards		conditional expectation	player's cards		conditional expectation
0	7	.588963	0	3	−.152118
1	6	.587416	1	2	−.152063
2	5	.588233	4	9	−.150726
3	4	.588793	5	8	−.150288
8	9	.588948	6	7	−.149122
0	6	.236674	0	2	−.190464
1	5	.236156	1	1	−.190411
2	4	.236708	3	9	−.189224
3	3	.236694	4	8	−.188670
7	9	.239118	5	7	−.192795
8	8	.237400	6	6	−.192720
0	5	.013429	0	1	−.215155
1	4	.007680	2	9	−.213829
2	3	.007477	3	8	−.218206
6	9	.015370	4	7	−.217162
7	8	.015797	5	6	−.216740
0	4	−.086007	0	0	−.228927
1	3	−.086453	1	9	−.232561
2	2	−.086333	2	8	−.232339
5	9	−.083974	3	7	−.230807
6	8	−.078498	4	6	−.230397
7	7	−.077890	5	5	−.230086

19.4 (a) 0.378868491; 0.117717739; 0.185726430; 0.317687341; 0.503413770; 0.435405080; 0.378868491; 0.303444169; 0.317687341; 4.938818850. (b) 0.378698225; 0.117642940; 0.185513866; 0.318144969; 0.503658835; 0.435787909; 0.378698225; 0.303156806; 0.318144969; 4.939446744. (c) $[83.207699421, 84.219958538]$; 0.202451823; 0.064409029; 83.620679032; 5.079781590.

19.5 H_0 for player: 241,149,546,272/19,524,993,263,685.
H for player: 7,535,923,321/552,096,050,907.
H_0 for banker: 114,753,351,728/10,847,218,479,825.
H for banker: 21,516,253,449/1,840,320,169,690.

$H_0 = H$ for tie: 103,841,353,768/723,147,898,655.

19.6 Yes, house advantage for player (resp., banker) bet appears to be decreasing (resp., increasing) in d.

19.7 (c) 0.005520; 0.005982. (d) 0.009661; 0.010424.

19.8 (a) 0.010521; 0.011746.

19.9 (a) player: 0.150967; 0.032001. banker: 0.270441; 0.032607. tie: 0.339027; 0.728289. (b) Maximal expectations are 2/3 for player, achieved by four 2s and two 3s, and by five 2s and one 3; 14/25 for banker, achieved by three 7s and three 8s; and 8 for tie, achieved by six cards of the same denomination, and by one 0 and five 3s.

19.10 −0.003662; −0.002137; 0.000663.

19.11 0: 0.236494; 1: 0.059599; 2: −0.110316; 3: −0.098707; 4: −0.134810; 5: −0.121918; 6: −0.534672; 7: −0.503260; 8: 0.301684; 9: 0.196424.

19.12 0–8: −.356745; 9: 4.280937. Point count is −1 for 0–8 and 12 for 9. 0.175602; 0.110791; 0.070441; 0.041746. (b) 0.222244; 0.109209; 0.072746; 0.043466.

19.13 0.012075017; 0.012076012.

19.14 −0.261; −0.039; 0.119; 0.210; 0.287; 0.007; −0.202; −0.141; −0.192; −0.234. Correlation is 0.946. $(-1, 0, 1, 2, 3, 1, -1, 0, -1, -1)$ has correlation 0.978.

19.15

19.16 See Table B.19.

Table B.19 A more accurate banker's secret strategy at baccara en banque (Problem 19.16).

	k										
j	0	1	2	3	4	5	6	7	8	9	∅
3	10.81	20.85	30.89	48.03	52.55	37.35	22.16	6.96	2.44	.77	49.38
4	14.43	4.39	5.65	15.70	32.83	37.35	22.16	6.96	8.23	17.64	49.38
5	32.83	29.62	19.58	9.54	.50	17.64	22.16	6.96	8.23	23.43	36.72
6	38.62	48.03	44.82	34.78	24.74	14.70	2.44	6.96	8.23	23.43	7.60

Chapter 20

20.1 31: 1,062,218,496; 32: 1,061,179,989; 33: 1,059,096,148; 34: 1,054,930,511; 35: 1,046,603,325; 36: 1,029,957,126; 37: 996,681,063; 38: 930,161,618; 39: 797,188,008; 40: 531,371,609.

20.2 (b) $[28.491292, 29.632797]$; 28.989182.

20.3 (a) 31: 0.148060863; 32: 0.137905177; 33: 0.127512672; 34: 0.116891073; 35: 0.106049464; 36: 0.094998365; 37: 0.083749795; 38: 0.072317327; 39: 0.060716146; 40: 0.051799118. (b) 34.603434712.

20.4 (a) 5.292290015.

20.5 See Table B.20. For the marginal total distribution, see Problem 20.3. For the marginal length distribution: 4: 0.262106; 5: 0.365194; 6: 0.238221; 7: $0.973343\cdot 10^{-01}$; 8: $0.288831\cdot 10^{-01}$; 9: $0.673199\cdot 10^{-02}$; 10: $0.128892\cdot 10^{-02}$; 11: $0.208348\cdot 10^{-03}$; 12: $0.289478\cdot 10^{-04}$; 13: $0.349883\cdot 10^{-05}$; 14: $0.370680\cdot 10^{-06}$; 15: $0.345630 \cdot 10^{-07}$; 16: $0.284046 \cdot 10^{-08}$; 17: $0.205675 \cdot 10^{-09}$; 18: $0.131011 \cdot 10^{-10}$; 19: $0.732277 \cdot 10^{-12}$; 20: $0.357930 \cdot 10^{-13}$; 21: $0.152306 \cdot 10^{-14}$; 22: $0.560865 \cdot 10^{-16}$; 23: $0.177354 \cdot 10^{-17}$; 24: $0.476632 \cdot 10^{-19}$; 25: $0.107371 \cdot 10^{-20}$; 26: $0.198965 \cdot 10^{-22}$; 27: $0.295342 \cdot 10^{-24}$; 28: $0.337646 \cdot 10^{-26}$; 29: $0.279104 \cdot 10^{-28}$; 30: $0.148474 \cdot 10^{-30}$; 31: $0.381680 \cdot 10^{-33}$.

20.6 31: 0.169905015; 32: 0.153373509; 33: 0.138153347; 34: 0.119652088; 35: 0.102287514; 36: 0.087618123; 37: 0.073512884; 38: 0.050131874; 39: 0.050139166; 40: 0.055226479.

20.8 0.011080; 0.012291.

20.9 $-.000931340$.

20.10 See Table B.21. 1; 0.175683.

20.11 (a) $\binom{24}{24}\binom{24}{3}\binom{24}{0}\cdots\binom{24}{0}\binom{96}{0}/\binom{312}{27} \approx 0.319192 \cdot 10^{-35}$. (b)

$$\sum_{m=0}^{12} \frac{\binom{24}{2m,24-2m,0}\binom{24}{15-m,3+m,6}\binom{24}{0,0,24}\cdots\binom{24}{0,0,24}\binom{96}{0,0,96}}{\binom{312}{15+m,27-m,270}} \approx 0.418326 \cdot 10^{-47}.$$

20.12 No.

20.13 -0.131526; 0.153417.

20.14 0.005976.

20.15 (a) 1/4. (c) Three black nines and five red tens yield 0.321429.

20.16 (a) -0.006379; -0.006604. (b) -0.006529; -0.006679. (c) -0.004961; -0.005130.

20.17 (a) Results are proportional to 53,200; 49,294; 45,785; 42,279; 39,044; 34,118; 31,104; 25,962; 22,150; 17,424. (b)

Chapter 21

21.1 (a) See the bottom row in Table B.22. (b) See Table B.22.

21.2 $N(0)$–$N(26)$: 1; 10; 55; 220; 715; 1,993; 4,915; 10,945; 22,330; 42,185; 74,396; 123,275; 192,950; 286,550; 405,350; 548,090; 710,675; 886,399; 1,066,715; 1,242,395; 1,404,815; 1,547,060; 1,664,582; 1,755,260; 1,818,905; 1,856,399; 1,868,755. Finally, $N(n) = N(52 - n)$ for $n = 27, 28, \ldots, 52$.

21.3 See Table B.23.

21.4 1: 0.013948; 2: 0.013735; 4: 0.013631; 6: 0.013597; 8: 0.013580.

21.5 See Table B.24.

Table B.20 Joint distribution of the total and length of a trente-et-quarante sequence, assuming sampling with replacement (Problem 20.5).

k	n	probability	n	probability	n	probability	n	probability	n	probability
4	31	$.534995 \cdot 10^{-01}$	32	$.454816 \cdot 10^{-01}$	33	$.382340 \cdot 10^{-01}$	34	$.317216 \cdot 10^{-01}$	35	$.259095 \cdot 10^{-01}$
5	31	$.517920 \cdot 10^{-01}$	32	$.500467 \cdot 10^{-01}$	33	$.474504 \cdot 10^{-01}$	34	$.441457 \cdot 10^{-01}$	35	$.402674 \cdot 10^{-01}$
6	31	$.285431 \cdot 10^{-01}$	32	$.281391 \cdot 10^{-01}$	33	$.276108 \cdot 10^{-01}$	34	$.268691 \cdot 10^{-01}$	35	$.258442 \cdot 10^{-01}$
7	31	$.105243 \cdot 10^{-01}$	32	$.105158 \cdot 10^{-01}$	33	$.104745 \cdot 10^{-01}$	34	$.103921 \cdot 10^{-01}$	35	$.102510 \cdot 10^{-01}$
8	31	$.291452 \cdot 10^{-02}$	32	$.292907 \cdot 10^{-02}$	33	$.294305 \cdot 10^{-02}$	34	$.295381 \cdot 10^{-02}$	35	$.295819 \cdot 10^{-02}$
9	31	$.646516 \cdot 10^{-03}$	32	$.651022 \cdot 10^{-03}$	33	$.656702 \cdot 10^{-03}$	34	$.663520 \cdot 10^{-03}$	35	$.671225 \cdot 10^{-03}$
10	31	$.119278 \cdot 10^{-03}$	32	$.120131 \cdot 10^{-03}$	33	$.121338 \cdot 10^{-03}$	34	$.122995 \cdot 10^{-03}$	35	$.125193 \cdot 10^{-03}$
11	31	$.187340 \cdot 10^{-04}$	32	$.188558 \cdot 10^{-04}$	33	$.190437 \cdot 10^{-04}$	34	$.193259 \cdot 10^{-04}$	35	$.197378 \cdot 10^{-04}$
12	31	$.254392 \cdot 10^{-05}$	32	$.255778 \cdot 10^{-05}$	33	$.258104 \cdot 10^{-05}$	34	$.261885 \cdot 10^{-05}$	35	$.267855 \cdot 10^{-05}$
13	31	$.301797 \cdot 10^{-06}$	32	$.303076 \cdot 10^{-06}$	33	$.305422 \cdot 10^{-06}$	34	$.309561 \cdot 10^{-06}$	35	$.316619 \cdot 10^{-06}$
14	31	$.314842 \cdot 10^{-07}$	32	$.315803 \cdot 10^{-07}$	33	$.317747 \cdot 10^{-07}$	34	$.321496 \cdot 10^{-07}$	35	$.328434 \cdot 10^{-07}$
15	31	$.289786 \cdot 10^{-08}$	32	$.290373 \cdot 10^{-08}$	33	$.291699 \cdot 10^{-08}$	34	$.294520 \cdot 10^{-08}$	35	$.300226 \cdot 10^{-08}$
16	31	$.235537 \cdot 10^{-09}$	32	$.235828 \cdot 10^{-09}$	33	$.236570 \cdot 10^{-09}$	34	$.238333 \cdot 10^{-09}$	35	$.242265 \cdot 10^{-09}$
17	31	$.168929 \cdot 10^{-10}$	32	$.169046 \cdot 10^{-10}$	33	$.169386 \cdot 10^{-10}$	34	$.170298 \cdot 10^{-10}$	35	$.172566 \cdot 10^{-10}$
18	31	$.106710 \cdot 10^{-11}$	32	$.106747 \cdot 10^{-11}$	33	$.106874 \cdot 10^{-11}$	34	$.107262 \cdot 10^{-11}$	35	$.108352 \cdot 10^{-11}$
19	31	$.592061 \cdot 10^{-13}$	32	$.592153 \cdot 10^{-13}$	33	$.592529 \cdot 10^{-13}$	34	$.593880 \cdot 10^{-13}$	35	$.598221 \cdot 10^{-13}$
20	31	$.287495 \cdot 10^{-14}$	32	$.287512 \cdot 10^{-14}$	33	$.287599 \cdot 10^{-14}$	34	$.287977 \cdot 10^{-14}$	35	$.289393 \cdot 10^{-14}$
21	31	$.121613 \cdot 10^{-15}$	32	$.121615 \cdot 10^{-15}$	33	$.121631 \cdot 10^{-15}$	34	$.121713 \cdot 10^{-15}$	35	$.122086 \cdot 10^{-15}$
22	31	$.445452 \cdot 10^{-17}$	32	$.445454 \cdot 10^{-17}$	33	$.445473 \cdot 10^{-17}$	34	$.445610 \cdot 10^{-17}$	35	$.446382 \cdot 10^{-17}$
23	31	$.140177 \cdot 10^{-18}$	32	$.140177 \cdot 10^{-18}$	33	$.140178 \cdot 10^{-18}$	34	$.140194 \cdot 10^{-18}$	35	$.140316 \cdot 10^{-18}$
24	31	$.375055 \cdot 10^{-20}$	32	$.375055 \cdot 10^{-20}$	33	$.375055 \cdot 10^{-20}$	34	$.375067 \cdot 10^{-20}$	35	$.375203 \cdot 10^{-20}$
25	31	$.841469 \cdot 10^{-22}$	32	$.841469 \cdot 10^{-22}$	33	$.841469 \cdot 10^{-22}$	34	$.841473 \cdot 10^{-22}$	35	$.841570 \cdot 10^{-22}$
26	31	$.155348 \cdot 10^{-23}$	32	$.155348 \cdot 10^{-23}$	33	$.155348 \cdot 10^{-23}$	34	$.155348 \cdot 10^{-23}$	35	$.155351 \cdot 10^{-23}$
27	31	$.229805 \cdot 10^{-25}$	32	$.229805 \cdot 10^{-25}$	33	$.229805 \cdot 10^{-25}$	34	$.229805 \cdot 10^{-25}$	35	$.229805 \cdot 10^{-25}$
28	31	$.261886 \cdot 10^{-27}$	32	$.261886 \cdot 10^{-27}$	33	$.261886 \cdot 10^{-27}$	34	$.261886 \cdot 10^{-27}$	35	$.261886 \cdot 10^{-27}$
29	31	$.215840 \cdot 10^{-29}$	32	$.215840 \cdot 10^{-29}$	33	$.215840 \cdot 10^{-29}$	34	$.215840 \cdot 10^{-29}$	35	$.215840 \cdot 10^{-29}$
30	31	$.114504 \cdot 10^{-31}$	32	$.114504 \cdot 10^{-31}$	33	$.114504 \cdot 10^{-31}$	34	$.114504 \cdot 10^{-31}$	35	$.114504 \cdot 10^{-31}$
31	31	$.293600 \cdot 10^{-34}$	32	$.293600 \cdot 10^{-34}$	33	$.293600 \cdot 10^{-34}$	34	$.293600 \cdot 10^{-34}$	35	$.293600 \cdot 10^{-34}$
4	36	$.207626 \cdot 10^{-01}$	37	$.162459 \cdot 10^{-01}$	38	$.123245 \cdot 10^{-01}$	39	$.896327 \cdot 10^{-02}$	40	$.896327 \cdot 10^{-02}$
5	36	$.359420 \cdot 10^{-01}$	37	$.312880 \cdot 10^{-01}$	38	$.264158 \cdot 10^{-01}$	39	$.214278 \cdot 10^{-01}$	40	$.164183 \cdot 10^{-01}$
6	36	$.244843 \cdot 10^{-01}$	37	$.227544 \cdot 10^{-01}$	38	$.206350 \cdot 10^{-01}$	39	$.181209 \cdot 10^{-01}$	40	$.152200 \cdot 10^{-01}$
7	36	$.100272 \cdot 10^{-01}$	37	$.969134 \cdot 10^{-02}$	38	$.921177 \cdot 10^{-02}$	39	$.855563 \cdot 10^{-02}$	40	$.769070 \cdot 10^{-02}$
8	36	$.295159 \cdot 10^{-02}$	37	$.292734 \cdot 10^{-02}$	38	$.287624 \cdot 10^{-02}$	39	$.278635 \cdot 10^{-02}$	40	$.264296 \cdot 10^{-02}$
9	36	$.679325 \cdot 10^{-03}$	37	$.686971 \cdot 10^{-03}$	38	$.692778 \cdot 10^{-03}$	39	$.694607 \cdot 10^{-03}$	40	$.689327 \cdot 10^{-03}$
10	36	$.128006 \cdot 10^{-03}$	37	$.131469 \cdot 10^{-03}$	38	$.135551 \cdot 10^{-03}$	39	$.140113 \cdot 10^{-03}$	40	$.144848 \cdot 10^{-03}$
11	36	$.203223 \cdot 10^{-04}$	37	$.211285 \cdot 10^{-04}$	38	$.222084 \cdot 10^{-04}$	39	$.236120 \cdot 10^{-04}$	40	$.253792 \cdot 10^{-04}$
12	36	$.277029 \cdot 10^{-05}$	37	$.290764 \cdot 10^{-05}$	38	$.310805 \cdot 10^{-05}$	39	$.339313 \cdot 10^{-05}$	40	$.378854 \cdot 10^{-05}$
13	36	$.328294 \cdot 10^{-06}$	37	$.347079 \cdot 10^{-06}$	38	$.376532 \cdot 10^{-06}$	39	$.421582 \cdot 10^{-06}$	40	$.488868 \cdot 10^{-06}$
14	36	$.340813 \cdot 10^{-07}$	37	$.362205 \cdot 10^{-07}$	38	$.398119 \cdot 10^{-07}$	39	$.456839 \cdot 10^{-07}$	40	$.550505 \cdot 10^{-07}$
15	36	$.311273 \cdot 10^{-08}$	37	$.331857 \cdot 10^{-08}$	38	$.368933 \cdot 10^{-08}$	39	$.433729 \cdot 10^{-08}$	40	$.543903 \cdot 10^{-08}$
16	36	$.250588 \cdot 10^{-09}$	37	$.267413 \cdot 10^{-09}$	38	$.300086 \cdot 10^{-09}$	39	$.361323 \cdot 10^{-09}$	40	$.472518 \cdot 10^{-09}$
17	36	$.177858 \cdot 10^{-10}$	37	$.189554 \cdot 10^{-10}$	38	$.214196 \cdot 10^{-10}$	39	$.263986 \cdot 10^{-10}$	40	$.360930 \cdot 10^{-10}$
18	36	$.111186 \cdot 10^{-11}$	37	$.118092 \cdot 10^{-11}$	38	$.133994 \cdot 10^{-11}$	39	$.168856 \cdot 10^{-11}$	40	$.242034 \cdot 10^{-11}$
19	36	$.610946 \cdot 10^{-13}$	37	$.645472 \cdot 10^{-13}$	38	$.733118 \cdot 10^{-13}$	39	$.943100 \cdot 10^{-13}$	40	$.142129 \cdot 10^{-12}$
20	36	$.294148 \cdot 10^{-14}$	37	$.308692 \cdot 10^{-14}$	38	$.349795 \cdot 10^{-14}$	39	$.458318 \cdot 10^{-14}$	40	$.728371 \cdot 10^{-14}$
21	36	$.123548 \cdot 10^{-15}$	37	$.128668 \cdot 10^{-15}$	38	$.144976 \cdot 10^{-15}$	39	$.192901 \cdot 10^{-15}$	40	$.324306 \cdot 10^{-15}$
22	36	$.450021 \cdot 10^{-17}$	37	$.464909 \cdot 10^{-17}$	38	$.519182 \cdot 10^{-17}$	39	$.698898 \cdot 10^{-17}$	40	$.124727 \cdot 10^{-16}$
23	36	$.141031 \cdot 10^{-18}$	37	$.144545 \cdot 10^{-18}$	38	$.159511 \cdot 10^{-18}$	39	$.216227 \cdot 10^{-18}$	40	$.411185 \cdot 10^{-18}$
24	36	$.376271 \cdot 10^{-20}$	37	$.382838 \cdot 10^{-20}$	38	$.416440 \cdot 10^{-20}$	39	$.565169 \cdot 10^{-20}$	40	$.115017 \cdot 10^{-19}$
25	36	$.842709 \cdot 10^{-22}$	37	$.852062 \cdot 10^{-22}$	38	$.911937 \cdot 10^{-22}$	39	$.123029 \cdot 10^{-21}$	40	$.269270 \cdot 10^{-21}$
26	36	$.155429 \cdot 10^{-23}$	37	$.156383 \cdot 10^{-23}$	38	$.164531 \cdot 10^{-23}$	39	$.218737 \cdot 10^{-23}$	40	$.517827 \cdot 10^{-23}$
27	36	$.229830 \cdot 10^{-25}$	37	$.230451 \cdot 10^{-25}$	38	$.238408 \cdot 10^{-25}$	39	$.309048 \cdot 10^{-25}$	40	$.796657 \cdot 10^{-25}$
28	36	$.261886 \cdot 10^{-27}$	37	$.262079 \cdot 10^{-27}$	38	$.267046 \cdot 10^{-27}$	39	$.333227 \cdot 10^{-27}$	40	$.942789 \cdot 10^{-27}$
29	36	$.215840 \cdot 10^{-29}$	37	$.215840 \cdot 10^{-29}$	38	$.217329 \cdot 10^{-29}$	39	$.257023 \cdot 10^{-29}$	40	$.805803 \cdot 10^{-29}$
30	36	$.114504 \cdot 10^{-31}$	37	$.114504 \cdot 10^{-31}$	38	$.114504 \cdot 10^{-31}$	39	$.125954 \cdot 10^{-31}$	40	$.442749 \cdot 10^{-31}$
31	36	$.293600 \cdot 10^{-34}$	37	$.293600 \cdot 10^{-34}$	38	$.293600 \cdot 10^{-34}$	39	$.293600 \cdot 10^{-34}$	40	$.117440 \cdot 10^{-33}$

21.6 Expectation is -0.005703880123. Total-dependent basic strategy under the assumptions of Baldwin et al. (1956):

Table B.21 Joint distribution of the total and length of a trente-et-quarante sequence (Problem 20.10).

k	n	probability	n	probability	n	probability	n	probability	n	probability
4	31	$.535835 \cdot 10^{-01}$	32	$.454004 \cdot 10^{-01}$	33	$.382147 \cdot 10^{-01}$	34	$.315609 \cdot 10^{-01}$	35	$.258280 \cdot 10^{-01}$
5	31	$.521412 \cdot 10^{-01}$	32	$.503734 \cdot 10^{-01}$	33	$.477568 \cdot 10^{-01}$	34	$.444029 \cdot 10^{-01}$	35	$.404907 \cdot 10^{-01}$
6	31	$.285617 \cdot 10^{-01}$	32	$.281977 \cdot 10^{-01}$	33	$.277042 \cdot 10^{-01}$	34	$.270034 \cdot 10^{-01}$	35	$.260055 \cdot 10^{-01}$
7	31	$.103221 \cdot 10^{-01}$	32	$.103554 \cdot 10^{-01}$	33	$.103546 \cdot 10^{-01}$	34	$.103094 \cdot 10^{-01}$	35	$.102044 \cdot 10^{-01}$
8	31	$.275633 \cdot 10^{-02}$	32	$.279169 \cdot 10^{-02}$	33	$.282425 \cdot 10^{-02}$	34	$.285180 \cdot 10^{-02}$	35	$.287128 \cdot 10^{-02}$
9	31	$.578449 \cdot 10^{-03}$	32	$.589969 \cdot 10^{-03}$	33	$.601343 \cdot 10^{-03}$	34	$.612823 \cdot 10^{-03}$	35	$.624454 \cdot 10^{-03}$
10	31	$.987283 \cdot 10^{-04}$	32	$.101377 \cdot 10^{-03}$	33	$.103911 \cdot 10^{-03}$	34	$.106523 \cdot 10^{-03}$	35	$.109399 \cdot 10^{-03}$
11	31	$.139725 \cdot 10^{-04}$	32	$.144593 \cdot 10^{-04}$	33	$.148956 \cdot 10^{-04}$	34	$.153316 \cdot 10^{-04}$	35	$.158217 \cdot 10^{-04}$
12	31	$.165792 \cdot 10^{-05}$	32	$.173208 \cdot 10^{-05}$	33	$.179353 \cdot 10^{-05}$	34	$.185132 \cdot 10^{-05}$	35	$.191564 \cdot 10^{-05}$
13	31	$.165781 \cdot 10^{-06}$	32	$.175272 \cdot 10^{-06}$	33	$.182520 \cdot 10^{-06}$	34	$.188792 \cdot 10^{-06}$	35	$.195525 \cdot 10^{-06}$
14	31	$.139645 \cdot 10^{-07}$	32	$.149894 \cdot 10^{-07}$	33	$.157133 \cdot 10^{-07}$	34	$.162786 \cdot 10^{-07}$	35	$.168466 \cdot 10^{-07}$
15	31	$.985383 \cdot 10^{-09}$	32	$.107847 \cdot 10^{-08}$	33	$.113980 \cdot 10^{-08}$	34	$.118242 \cdot 10^{-08}$	35	$.122103 \cdot 10^{-08}$
16	31	$.576683 \cdot 10^{-10}$	32	$.647053 \cdot 10^{-10}$	33	$.690752 \cdot 10^{-10}$	34	$.717634 \cdot 10^{-10}$	35	$.738697 \cdot 10^{-10}$
17	31	$.276118 \cdot 10^{-11}$	32	$.319752 \cdot 10^{-11}$	33	$.345511 \cdot 10^{-11}$	34	$.359552 \cdot 10^{-11}$	35	$.368710 \cdot 10^{-11}$
18	31	$.106333 \cdot 10^{-12}$	32	$.128138 \cdot 10^{-12}$	33	$.140441 \cdot 10^{-12}$	34	$.146395 \cdot 10^{-12}$	35	$.149532 \cdot 10^{-12}$
19	31	$.322483 \cdot 10^{-14}$	32	$.408552 \cdot 10^{-14}$	33	$.455071 \cdot 10^{-14}$	34	$.474986 \cdot 10^{-14}$	35	$.483289 \cdot 10^{-14}$
20	31	$.750071 \cdot 10^{-16}$	32	$.101209 \cdot 10^{-15}$	33	$.114762 \cdot 10^{-15}$	34	$.119828 \cdot 10^{-15}$	35	$.121475 \cdot 10^{-15}$
21	31	$.129277 \cdot 10^{-17}$	32	$.188983 \cdot 10^{-17}$	33	$.218420 \cdot 10^{-17}$	34	$.227784 \cdot 10^{-17}$	35	$.230114 \cdot 10^{-17}$
22	31	$.157679 \cdot 10^{-19}$	32	$.255572 \cdot 10^{-19}$	33	$.301223 \cdot 10^{-19}$	34	$.313068 \cdot 10^{-19}$	35	$.315255 \cdot 10^{-19}$
23	31	$.127686 \cdot 10^{-21}$	32	$.237097 \cdot 10^{-21}$	33	$.284697 \cdot 10^{-21}$	34	$.294149 \cdot 10^{-21}$	35	$.295372 \cdot 10^{-21}$
24	31	$.625511 \cdot 10^{-24}$	32	$.139722 \cdot 10^{-23}$	33	$.170296 \cdot 10^{-23}$	34	$.174523 \cdot 10^{-23}$	35	$.174870 \cdot 10^{-23}$
25	31	$.160231 \cdot 10^{-26}$	32	$.465836 \cdot 10^{-26}$	33	$.571538 \cdot 10^{-26}$	34	$.580366 \cdot 10^{-26}$	35	$.580757 \cdot 10^{-26}$
26	31	$.166638 \cdot 10^{-29}$	32	$.726117 \cdot 10^{-29}$	33	$.884076 \cdot 10^{-29}$	34	$.890511 \cdot 10^{-29}$	35	$.890629 \cdot 10^{-29}$
27	31	$.412628 \cdot 10^{-33}$	32	$.370069 \cdot 10^{-32}$	33	$.439521 \cdot 10^{-32}$	34	$.440450 \cdot 10^{-32}$	35	$.440453 \cdot 10^{-32}$
28	31	$0.000000 \cdot 10^{+00}$	32	$.235194 \cdot 10^{-36}$	33	$.268793 \cdot 10^{-36}$	34	$.268793 \cdot 10^{-36}$	35	$.268793 \cdot 10^{-36}$
4	36	$.205556 \cdot 10^{-01}$	37	$.161382 \cdot 10^{-01}$	38	$.121041 \cdot 10^{-01}$	39	$.885439 \cdot 10^{-02}$	40	$.857769 \cdot 10^{-02}$
5	36	$.361045 \cdot 10^{-01}$	37	$.314125 \cdot 10^{-01}$	38	$.264846 \cdot 10^{-01}$	39	$.214676 \cdot 10^{-01}$	40	$.164156 \cdot 10^{-01}$
6	36	$.246738 \cdot 10^{-01}$	37	$.229517 \cdot 10^{-01}$	38	$.208373 \cdot 10^{-01}$	39	$.183062 \cdot 10^{-01}$	40	$.153826 \cdot 10^{-01}$
7	36	$.100124 \cdot 10^{-01}$	37	$.970591 \cdot 10^{-02}$	38	$.924926 \cdot 10^{-02}$	39	$.861065 \cdot 10^{-02}$	40	$.775481 \cdot 10^{-02}$
8	36	$.287872 \cdot 10^{-02}$	37	$.286764 \cdot 10^{-02}$	38	$.282937 \cdot 10^{-02}$	39	$.275192 \cdot 10^{-02}$	40	$.262058 \cdot 10^{-02}$
9	36	$.636002 \cdot 10^{-03}$	37	$.646905 \cdot 10^{-03}$	38	$.656047 \cdot 10^{-03}$	39	$.661570 \cdot 10^{-03}$	40	$.660594 \cdot 10^{-03}$
10	36	$.112702 \cdot 10^{-03}$	37	$.116556 \cdot 10^{-03}$	38	$.121025 \cdot 10^{-03}$	39	$.126068 \cdot 10^{-03}$	40	$.131490 \cdot 10^{-03}$
11	36	$.164259 \cdot 10^{-04}$	37	$.172089 \cdot 10^{-04}$	38	$.182393 \cdot 10^{-04}$	39	$.195862 \cdot 10^{-04}$	40	$.213125 \cdot 10^{-04}$
12	36	$.199844 \cdot 10^{-05}$	37	$.211431 \cdot 10^{-05}$	38	$.228133 \cdot 10^{-05}$	39	$.252173 \cdot 10^{-05}$	40	$.286248 \cdot 10^{-05}$
13	36	$.204460 \cdot 10^{-06}$	37	$.217857 \cdot 10^{-06}$	38	$.238783 \cdot 10^{-06}$	39	$.271470 \cdot 10^{-06}$	40	$.321725 \cdot 10^{-06}$
14	36	$.176148 \cdot 10^{-07}$	37	$.188510 \cdot 10^{-07}$	38	$.209463 \cdot 10^{-07}$	39	$.244898 \cdot 10^{-07}$	40	$.303671 \cdot 10^{-07}$
15	36	$.127339 \cdot 10^{-08}$	37	$.136444 \cdot 10^{-08}$	38	$.153332 \cdot 10^{-08}$	39	$.184403 \cdot 10^{-08}$	40	$.240116 \cdot 10^{-08}$
16	36	$.766635 \cdot 10^{-10}$	37	$.819636 \cdot 10^{-10}$	38	$.928547 \cdot 10^{-10}$	39	$.114860 \cdot 10^{-09}$	40	$.157818 \cdot 10^{-09}$
17	36	$.380174 \cdot 10^{-11}$	37	$.404177 \cdot 10^{-11}$	38	$.459654 \cdot 10^{-11}$	39	$.584276 \cdot 10^{-11}$	40	$.851900 \cdot 10^{-11}$
18	36	$.153066 \cdot 10^{-12}$	37	$.161347 \cdot 10^{-12}$	38	$.183271 \cdot 10^{-12}$	39	$.238841 \cdot 10^{-12}$	40	$.371837 \cdot 10^{-12}$
19	36	$.491213 \cdot 10^{-14}$	37	$.512378 \cdot 10^{-14}$	38	$.578030 \cdot 10^{-14}$	39	$.769217 \cdot 10^{-14}$	40	$.128729 \cdot 10^{-13}$
20	36	$.122711 \cdot 10^{-15}$	37	$.126567 \cdot 10^{-15}$	38	$.140999 \cdot 10^{-15}$	39	$.190421 \cdot 10^{-15}$	40	$.345068 \cdot 10^{-15}$
21	36	$.231373 \cdot 10^{-17}$	37	$.236121 \cdot 10^{-17}$	38	$.258417 \cdot 10^{-17}$	39	$.351044 \cdot 10^{-17}$	40	$.694420 \cdot 10^{-17}$
22	36	$.316021 \cdot 10^{-19}$	37	$.319678 \cdot 10^{-19}$	38	$.342457 \cdot 10^{-19}$	39	$.462294 \cdot 10^{-19}$	40	$.100706 \cdot 10^{-18}$
23	36	$.295615 \cdot 10^{-21}$	37	$.297187 \cdot 10^{-21}$	38	$.311274 \cdot 10^{-21}$	39	$.411124 \cdot 10^{-21}$	40	$.995220 \cdot 10^{-21}$
24	36	$.174903 \cdot 10^{-23}$	37	$.175218 \cdot 10^{-23}$	38	$.179826 \cdot 10^{-23}$	39	$.228183 \cdot 10^{-23}$	40	$.618896 \cdot 10^{-23}$
25	36	$.580771 \cdot 10^{-26}$	37	$.580992 \cdot 10^{-26}$	38	$.587411 \cdot 10^{-26}$	39	$.703244 \cdot 10^{-26}$	40	$.214741 \cdot 10^{-25}$
26	36	$.890630 \cdot 10^{-29}$	37	$.890661 \cdot 10^{-29}$	38	$.893268 \cdot 10^{-29}$	39	$.997780 \cdot 10^{-29}$	40	$.341489 \cdot 10^{-28}$
27	36	$.440453 \cdot 10^{-32}$	37	$.440453 \cdot 10^{-32}$	38	$.440569 \cdot 10^{-32}$	39	$.461085 \cdot 10^{-32}$	40	$.173410 \cdot 10^{-31}$
28	36	$.268793 \cdot 10^{-36}$	37	$.268793 \cdot 10^{-36}$	38	$.268793 \cdot 10^{-36}$	39	$.268793 \cdot 10^{-36}$	40	$.107517 \cdot 10^{-35}$

1. *Insurance and dealer natural.* Is dealer's upcard an ace or a 10-valued card? If not, go to Step 2.

   - If dealer's upcard is an ace, do not take insurance.

   If dealer has a natural, stop. Otherwise go to Step 2.
2. *Splitting.* Does hand consist of a pair? If not, go to Step 3.

Table B.22 The number of distinct blackjack hands with hard total m and size n (Problem 21.1).

					n						
m	2	3	4	5	6	7	8	9	10	11	total
21	0	12	41	74	89	82	54	26	7	1	386
20	1	13	41	65	76	65	41	17	5	0	324
19	1	14	38	58	62	51	28	11	2	0	265
18	2	15	36	50	52	38	20	7	1	0	221
17	2	15	32	43	40	28	13	4	0	0	177
16	3	15	30	35	32	20	9	2	0	0	146
15	3	15	25	28	24	14	5	1	0	0	115
14	4	14	22	23	18	9	3	0	0	0	93
13	4	13	18	18	12	6	1	0	0	0	72
12	5	12	15	13	9	3	1	0	0	0	58
11	5	10	11	10	5	2	0	0	0	0	43
10	5	8	9	6	4	1	0	0	0	0	33
9	4	7	6	5	2	0	0	0	0	0	24
8	4	5	5	3	1	0	0	0	0	0	18
7	3	4	3	2	0	0	0	0	0	0	12
6	3	3	2	1	0	0	0	0	0	0	9
5	2	2	1	0	0	0	0	0	0	0	5
4	2	1	1	0	0	0	0	0	0	0	4
3	1	1	0	0	0	0	0	0	0	0	2
2	1	0	0	0	0	0	0	0	0	0	1
total	55	179	336	434	426	319	175	68	15	1	2,008

- Always split $\{A, A\}$ and $\{8, 8\}$. Never split $\{5, 5\}$ or $\{T, T\}$. Split $\{2, 2\}$, $\{3, 3\}$, and $\{7, 7\}$ vs. 2–7, $\{4, 4\}$ vs. 5–6, $\{6, 6\}$ vs. 2–6, and $\{9, 9\}$ vs. 2–9 except 7.

If aces are split, stop. If any other pair is split, apply this algorithm, beginning with Step 3, to each hand. Otherwise go to Step 3.

3. *Doubling*. Does hand consist of the initial two cards only (possibly after a split)? If not, go to Step 4.

- *Hard totals*. Double 11 vs. 2–T. Double 10 vs. 2–9. Double 9 vs. 3–6.
- *Soft totals*. Double 13 vs. 6, 14–15 vs. 5–6, 16 vs. 4–6, and 17–18 vs. 3–6.

If hand is doubled, stop. Otherwise go to Step 4.

4. *Hitting and standing*.

Table B.23 Player's cvonditional expectation given player's abstract total and dealer's upcard (Problem 21.3). (Rules: See Table 21.1.)

up-		player's abstract total				
card	stiff	17	18	19	20	21*
2	−.294 055	−.155 079	.115 659	.379 237	.635 000	.879 474
3	−.248 824	−.118 511	.142 749	.397 456	.644 562	.883 953
4	−.194 394	−.063 421	.181 714	.416 556	.653 521	.884 904
5	−.142 190	−.022 502	.220 668	.461 060	.682 664	.893 679
6	−.158 354	.008 594	.281 995	.495 641	.703 538	.902 122
7	−.480 293	−.107 948	.402 980	.618 898	.775 129	.927 013
8	−.522 745	−.391 888	.101 959	.594 393	.792 127	.930 209
9	−.533 115	−.411 229	−.185 422	.275 891	.755 532	.938 891
T	−.535 002	−.410 846	−.164 204	.082 702	.563 992	.960 430
A	−.660 370	−.476 584	−.101 908	.277 662	.658 034	.924 863

*nonnatural

Table B.24 Conditional distribution of dealer's final total, given dealer's upcard. Assumes sampling with replacement (Problem 21.5). (Rules: See Table 21.1)

up-				dealer's final total			
card	17	18	19	20	21*	21**	bust
2	.139 809	.134 907	.129 655	.124 026	.117 993		.353 608
3	.135 034	.130 482	.125 581	.120 329	.114 700		.373 875
4	.130 490	.125 938	.121 386	.116 485	.111 233		.394 468
5	.122 251	.122 251	.117 700	.113 148	.108 246		.416 404
6	.165 438	.106 267	.106 267	.101 715	.097 163		.423 150
7	.368 566	.137 797	.078 625	.078 625	.074 074		.262 312
8	.128 567	.359 336	.128 567	.069 395	.069 395		.244 741
9	.119 995	.119 995	.350 765	.119 995	.060 824		.228 425
T	.111 424	.111 424	.111 424	.342 194	.034 501	.076 923	.212 109
A	.130 789	.130 789	.130 789	.130 789	.053 866	.307 692	.115 286

*three or more cards **two cards (natural)

- *Hard totals.* Always stand on 17 or higher. Hit stiffs (12–16) vs. high cards $(7, 8, 9, \mathrm{T}, \mathrm{A})$. Stand on stiffs vs. low cards $(2, 3, 4, 5, 6)$, except hit 12 vs. 2 and 3. Always hit 11 or lower.

- *Soft totals*. Hit 17 or less, and stand on 18 or more, except hit 18 vs. 9, T, A.

After standing or busting, stop. After hitting without busting, repeat Step 4.

21.7 stand: 0.482353900; hit: 0.241312966; double: 0.482625931.
21.8 stand: 0.697402790; split: 0.520374382.
21.9 See Table B.25. The closest decision is $\{2, 6\}$ vs. 5.

Table B.25 Composition-dependent basic strategy practice hands (Problem 21.9). (Rules: See Table 21.1.)

hand	stand	hit	double	split
77 vs. T	−0.509739350541	−0.514818210092	−1.034723820806	(1)
49 vs. 2	−0.285725661496	−0.293007593298	−0.586015186595	
66 vs. 7	−0.493436956643	−0.264853865729	−0.598513511182	(2)
39 vs. 3	−0.261815171261	−0.255711843192	−0.511423686384	
2T vs. 6	−0.160378641754	−0.159435817659	−0.318871635318	
44 vs. 5	−0.094896587995	0.153926501483	0.162314209377	(3)
26 vs. 5	−0.103818941813	0.130629787231	0.130582861892	
A8 vs. 6	0.460791633862	0.226529289550	0.453058579101	
A6 vs. 2	−0.131767396493	0.007097780732	0.013320956371	
A2 vs. 4	−0.186079424489	0.110212986461	0.115097236657	
22246 vs. 9	−0.531631763142	−0.531718245721		
A23T vs. T	−0.542177311306	−0.542257615484		
AA2228 vs. 7	−0.450569411261	−0.450718982494		
AAAA2226 vs. T	−0.559354995543	−0.559438727278		
AA256 vs. T	−0.554071521619	−0.553787181209		
A2T vs. 2	−0.300727949432	−0.300667653748		
A236 vs. 3	−0.222724028368	−0.222632517486		
AAAA26 vs. 2	−0.234752434809	−0.235442446883		
AA6 vs. A	−0.132828588996	−0.132219688132		
AAAA22 vs. T	−0.190932581630	−0.189646097689		

(1) −0.636691424373; (2) −0.267864128552; (3) 0.090804010400.

21.10 To double $\{T, T\}$ vs. 8 costs about 2.468706 in expectation.
21.11

```
          stand            hit          double           split
0000000002 hard 20
  1   0.650096886072  -0.883183100218  -1.766366200435  -0.354749276413  stand
  2   0.627225893166  -0.846660804406  -1.693321608812   0.048114912609  stand
  3   0.636133985311  -0.846301142159  -1.692602284319   0.124252927682  stand
  4   0.644848455237  -0.846272556038  -1.692545112077   0.224812689909  stand
  5   0.673675300425  -0.845596878138  -1.691193756277   0.326617820316  stand
  6   0.697402789715  -0.845026577396  -1.690053154793   0.363571294016  stand
  7   0.764676549258  -0.843026011444  -1.686052022887   0.251782528630  stand
  8   0.783250887195  -0.842727419697  -1.685454839393   0.011765187565  stand
  9   0.743970134237  -0.842055464844  -1.684110929688  -0.255830042716  stand
 10   0.583153676761  -0.836968970204  -1.673937940407  -0.316427570405  stand
0000000011 hard 19
  1   0.307676129461  -0.742644805033  -1.485289610067  stand
  2   0.384834065698  -0.749660654740  -1.499321309479  stand
  3   0.383556602029  -0.712571744993  -1.425143489986  stand
```

```
  4   0.404114288636  -0.711695628353  -1.423391256706  stand
  5   0.447849250479  -0.708740231017  -1.417480462033  stand
  6   0.484092892922  -0.706989091184  -1.413978182368  stand
  7   0.610120488875  -0.698545868747  -1.397091737495  stand
  8   0.576828276389  -0.697330566606  -1.394661133211  stand
  9   0.264278858742  -0.697795043762  -1.395590087523  stand
 10   0.102517351540  -0.710036528083  -1.420073056167  stand
0000000020 hard 18
  1  -0.055173687540  -0.624843140684  -1.249686281367  -0.071658866588  stand
  2   0.137057412492  -0.627497078738  -1.254994157475   0.172932952303  split
  3   0.122552885433  -0.638280860167  -1.276561720334   0.172687150684  split
  4   0.166978330894  -0.597001221214  -1.194002442427   0.258737091751  split
  5   0.202892594601  -0.590310217041  -1.180620434083   0.349718604530  split
  6   0.265195246753  -0.586712793478  -1.173425586956   0.365949715938  split
  7   0.401060096346  -0.566048547786  -1.132097095573   0.334968796619  stand
  8   0.064517870144  -0.566264774510  -1.132529549020   0.190275558063  split
  9  -0.196371805808  -0.594732336125  -1.189464672250  -0.108825178752  split
 10  -0.133284961860  -0.624464723391  -1.248929446782  -0.277464271631  stand
0000000101 hard 18
  1  -0.082020292286  -0.633177454024  -1.266354908049  stand
  2   0.118877005022  -0.632537306211  -1.265074612422  stand
  3   0.144413719791  -0.633653293590  -1.267306587181  stand
  4   0.164239541043  -0.597316713220  -1.194633426441  stand
  5   0.202289626141  -0.590755334476  -1.181510668953  stand
  6   0.268100509999  -0.586114838750  -1.172229677501  stand
  7   0.388745975508  -0.567258622824  -1.134517245647  stand
  8   0.095529719842  -0.565136966001  -1.130273932003  stand
  9  -0.196135818721  -0.593226948278  -1.186453896555  stand
 10  -0.155191975257  -0.622794459232  -1.245588918464  stand
0000000110 hard 17
  1  -0.451875212887  -0.544608360419  -1.089216720839  stand
  2  -0.136521360357  -0.530170775607  -1.060341551214  stand
  3  -0.120663616868  -0.535485754939  -1.070971509878  stand
  4  -0.084413653561  -0.541139535427  -1.082279070853  stand
  5  -0.044394196282  -0.492581822189  -0.985163644379  stand
  6  -0.011411359131  -0.483777217239  -0.967554434479  stand
  7  -0.122899621021  -0.448819285920  -0.897638571840  stand
  8  -0.414898746317  -0.475296409004  -0.950592818009  stand
  9  -0.411645547968  -0.531881894895  -1.063763789789  stand
 10  -0.390690687323  -0.558164886786  -1.116329773572  stand
0000001001 hard 17
  1  -0.467040735699  -0.555801946420  -1.111603892840  stand
  2  -0.158128238098  -0.538453449695  -1.076906899391  stand
  3  -0.118942533932  -0.536375251339  -1.072750502677  stand
  4  -0.064395094831  -0.535107962834  -1.070215925668  stand
  5  -0.043147604946  -0.492433646775  -0.984867293550  stand
  6  -0.011286653066  -0.483342785468  -0.966685570936  stand
  7  -0.121287404918  -0.451962685507  -0.903925371015  stand
  8  -0.394239912324  -0.473802501094  -0.947605002189  stand
  9  -0.416111157832  -0.526459140476  -1.052918280952  stand
 10  -0.412340737003  -0.556703733990  -1.113407467979  stand
0000000200 hard 16
  1  -0.643531885807  -0.494905391674  -0.989810783348  -0.324889036918  split
  2  -0.274813531019  -0.454093200240  -0.908186400479   0.043727194275  split
  3  -0.228353511464  -0.449931475392  -0.899862950785   0.110720775722  split
  4  -0.215265876458  -0.461145695126  -0.922291390251   0.133230490961  split
  5  -0.165443338329  -0.452874351922  -0.905748703844   0.218373360417  split
  6  -0.178170510118  -0.396691845440  -0.793383690879   0.269512766883  split
  7  -0.502514243121  -0.373560953395  -0.747121906790   0.251583640976  split
  8  -0.551275826794  -0.426315012550  -0.852630025100  -0.086970515923  split
  9  -0.516426474719  -0.487123932320  -0.974247864640  -0.426300902131  split
 10  -0.518291103034  -0.511755197679  -1.023510395358  -0.457866113246  split
0000001010 hard 16
  1  -0.643246505943  -0.495493307394  -0.990986614788  hit
  2  -0.276548296509  -0.455974342267  -0.911948684535  stand
  3  -0.249700055558  -0.459005852326  -0.918011704652  stand
  4  -0.194143246184  -0.452965189418  -0.905930378837  stand
  5  -0.163593463883  -0.452056919901  -0.904113839802  stand
  6  -0.179634154472  -0.397209847572  -0.794419695145  stand
  7  -0.507326854118  -0.374945053600  -0.749890107199  hit
  8  -0.525806002725  -0.427790260794  -0.855580521589  hit
  9  -0.539835554663  -0.482037461884  -0.964074923768  hit
 10  -0.518034139317  -0.512008735213  -1.024017470426  hit
0000010001 hard 16
  1  -0.652908972348  -0.508750150191  -1.017500300382  hit
  2  -0.297664407112  -0.465395637522  -0.930791275044  stand
  3  -0.249875554484  -0.461401514826  -0.922803029651  stand
  4  -0.193441780066  -0.454566338567  -0.909132677134  stand
  5  -0.141194472764  -0.444223030076  -0.888446060153  stand
  6  -0.179022898802  -0.396277569046  -0.792555138093  stand
  7  -0.483324420261  -0.376193509207  -0.752387018413  hit
  8  -0.527006523861  -0.424822549943  -0.849645099887  hit
  9  -0.539231513534  -0.479306375180  -0.958612750359  hit
 10  -0.542951853825  -0.506929242579  -1.013858485158  hit
0000001100 hard 15
  1  -0.637157778909  -0.455655642791  -0.928014910412  hit
  2  -0.272058235456  -0.390377438794  -0.780754877588  stand
  3  -0.223853651977  -0.381275859166  -0.762551718331  stand
  4  -0.189589586916  -0.380044419986  -0.760088839973  stand
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  5  -0.160466525333  -0.373118334088  -0.746236668177  stand
  6  -0.176975891044  -0.362115861247  -0.724231722493  stand
  7  -0.501385013922  -0.324113643717  -0.666031053936  hit
  8  -0.545307975577  -0.379554352354  -0.777379128551  hit
  9  -0.535908729539  -0.443182115572  -0.893558162982  hit
 10  -0.514015226787  -0.474793692804  -0.949587385607  hit
0000010010 hard 15
  1  -0.636808706185  -0.495852837663  -1.008370006355  hit
  2  -0.273298554472  -0.430256697288  -0.860513394575  stand
  3  -0.245986549270  -0.432276340155  -0.864552680310  stand
  4  -0.188621770039  -0.419615391455  -0.839230782909  stand
  5  -0.136854617797  -0.402035178898  -0.804070357796  stand
  6  -0.176502556715  -0.400876273387  -0.801752546774  stand
  7  -0.502845272080  -0.363207713665  -0.744596192190  hit
  8  -0.523098577703  -0.420732665189  -0.855589063436  hit
  9  -0.535540675497  -0.479660916111  -0.966804469747  hit
 10  -0.539248150121  -0.509826715066  -1.023930499627  hit
0000100001 hard 15
  1  -0.669157224799  -0.498740752611  -1.012900150676  hit
  2  -0.294782706444  -0.436218538366  -0.872437076731  stand
  3  -0.247377792507  -0.429757987838  -0.859515975677  stand
  4  -0.190686829814  -0.417990075414  -0.835980150829  stand
  5  -0.135682746571  -0.399583508742  -0.799167017485  stand
  6  -0.154176471950  -0.387326756859  -0.774653513718  stand
  7  -0.478497330733  -0.364502921604  -0.736264938036  hit
  8  -0.524234203444  -0.417965202405  -0.844148481650  hit
  9  -0.535092786309  -0.475256276227  -0.951923864815  hit
 10  -0.538432812034  -0.501091449795  -1.002182899590  hit
0000002000 hard 14
  1  -0.630501705653  -0.494721083279  -1.029342617741  -0.611817514697  hit
  2  -0.268308533216  -0.406387852911  -0.812775705823  -0.151896350303  split
  3  -0.219400379679  -0.388265151945  -0.776530303890  -0.063941323594  split
  4  -0.163936581432  -0.368802577406  -0.737605154812   0.034107273409  split
  5  -0.155508591303  -0.370326658248  -0.740653316496   0.055776850293  split
  6  -0.174225330443  -0.366940958705  -0.733881917410   0.072757499210  split
  7  -0.501956464996  -0.389227188689  -0.823012449019  -0.110319494833  split
  8  -0.539340124360  -0.407892515444  -0.857864974809  -0.422273571453  hit
  9  -0.555390984359  -0.474654103466  -0.978237704211  -0.568857383751  hit
 10  -0.509739350541  -0.514818210092  -1.034723820806  -0.636691424373  stand
0000010100 hard 14
  1  -0.630436299702  -0.453389421101  -0.941763068665  hit
  2  -0.267800089608  -0.362615443866  -0.725230887733  stand
  3  -0.220210441236  -0.346968862594  -0.693937725187  stand
  4  -0.184078389319  -0.337541603198  -0.675083206395  stand
  5  -0.133746558213  -0.317191842760  -0.634383685520  stand
  6  -0.172288351761  -0.323254032917  -0.646508065833  stand
  7  -0.498604112157  -0.348533078430  -0.734594078763  hit
  8  -0.542600550555  -0.369091538533  -0.774544862835  hit
  9  -0.531613850373  -0.437186843584  -0.887880746437  hit
 10  -0.535229237591  -0.464761489361  -0.936341224645  hit
0000100010 hard 14
  1  -0.653430355014  -0.441214984119  -0.920604700136  hit
  2  -0.269314207717  -0.358925914389  -0.717851828778  stand
  3  -0.243557609709  -0.356044617672  -0.712089235344  stand
  4  -0.185886328053  -0.335517930827  -0.671035861654  stand
  5  -0.131352331087  -0.313835150303  -0.627670300605  stand
  6  -0.150100188338  -0.308744178299  -0.617488356598  stand
  7  -0.499718862824  -0.348014834326  -0.727640877381  hit
  8  -0.520326257286  -0.370138613176  -0.761927893280  hit
  9  -0.531401948271  -0.425857937945  -0.859832880875  hit
 10  -0.534729108331  -0.453066233021  -0.911135442845  hit
0001000001 hard 14
  1  -0.670237028296  -0.444960621849  -0.931613153749  hit
  2  -0.310107308585  -0.368778115579  -0.737556231158  stand
  3  -0.250869576518  -0.355683075981  -0.711366151963  stand
  4  -0.193331420653  -0.336458527940  -0.672917055880  stand
  5  -0.138176503403  -0.314638308241  -0.629276616482  stand
  6  -0.155348725082  -0.308093142486  -0.616186284971  stand
  7  -0.468887369236  -0.342198856909  -0.706807577216  hit
  8  -0.523849635073  -0.357370102393  -0.739261685781  hit
  9  -0.535907728382  -0.413519784968  -0.839393541851  hit
 10  -0.539296210123  -0.445788392047  -0.899255609252  hit
0000011000 hard 13
  1  -0.623767218739  -0.443346811368  -0.937178324988  hit
  2  -0.265045618697  -0.331966083271  -0.663932166542  stand
  3  -0.214770943146  -0.304029325852  -0.608058651705  stand
  4  -0.158371702733  -0.277039658086  -0.554079316173  stand
  5  -0.128820928194  -0.269028409162  -0.538056818323  stand
  6  -0.169691969390  -0.280666452521  -0.561332905042  stand
  7  -0.497619621705  -0.330722947136  -0.714108562084  hit
  8  -0.538333379611  -0.394373090850  -0.842908011913  hit
  9  -0.551096105193  -0.418484156464  -0.869389577364  hit
 10  -0.530953361345  -0.450337085148  -0.908269746695  hit
0000100100 hard 13
  1  -0.647201483402  -0.433029817590  -0.930925277637  hit
  2  -0.264894904207  -0.331281381381  -0.662562762761  stand
  3  -0.216700170705  -0.305572354098  -0.611144708197  stand
  4  -0.181301847441  -0.291820186334  -0.583640372668  stand
  5  -0.128263570003  -0.270014665388  -0.540029330776  stand
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  6  -0.146040161614  -0.267603324227  -0.535206648453  stand
  7  -0.493921761375  -0.327475020847  -0.706532474765  hit
  8  -0.541528910410  -0.386852289748  -0.829133846295  hit
  9  -0.527475123147  -0.412582321130  -0.847601997898  hit
 10  -0.530710195801  -0.446362339053  -0.909613616269  hit
0001000010 hard 13
  1  -0.654533535693  -0.383074934588  -0.838019069134  hit
  2  -0.285725661496  -0.293007593298  -0.586015186595  stand
  3  -0.245979270944  -0.280361097749  -0.560722195498  stand
  4  -0.188472708744  -0.254282765395  -0.508565530791  stand
  5  -0.133858673897  -0.232059061759  -0.464118123518  stand
  6  -0.151344810842  -0.228133983707  -0.456267967415  stand
  7  -0.489330930565  -0.274092235083  -0.600139861062  hit
  8  -0.520792029052  -0.338911134356  -0.724110253228  hit
  9  -0.532216890344  -0.365184149354  -0.758677008769  hit
 10  -0.535592506419  -0.398701717434  -0.818900733566  hit
0010000001 hard 13
  1  -0.672168144341  -0.392502663572  -0.859048823167  hit
  2  -0.312389646126  -0.304214648881  -0.608429297762  hit
  3  -0.265647782712  -0.283224158301  -0.566448316601  stand
  4  -0.197108745306  -0.257952202368  -0.515904404735  stand
  5  -0.141140766921  -0.235302804135  -0.470605608270  stand
  6  -0.157772688978  -0.228909363855  -0.457818727710  stand
  7  -0.471067373042  -0.270391240796  -0.582990027327  hit
  8  -0.515178692416  -0.328234857003  -0.705639998322  hit
  9  -0.536257079851  -0.358831796967  -0.752137667230  hit
 10  -0.539001558394  -0.393091074235  -0.811084292236  hit
0000020000 hard 12
  1  -0.617010348043  -0.386203662772  -0.834601543265  -0.658601943124  hit
  2  -0.261713595940  -0.252670853654  -0.505341707309  -0.212364607351  split
  3  -0.211033361962  -0.222133678184  -0.444267356369  -0.121629489752  split
  4  -0.151882717672  -0.190113403769  -0.380226807539  -0.014073802832  split
  5  -0.102165797930  -0.162359809986  -0.324719619971   0.085134021345  split
  6  -0.165187346321  -0.193568437678  -0.387136875357  -0.003899295615  split
  7  -0.493436956643  -0.264853865729  -0.598513511182  -0.267864128552  hit
  8  -0.535770693336  -0.321706657801  -0.711353503377  -0.418613262912  hit
  9  -0.548501906299  -0.386241735506  -0.817364163634  -0.584835427577  hit
 10  -0.552167372149  -0.386149115666  -0.803851881727  -0.671579009155  hit
0000101000 hard 12
  1  -0.640619164528  -0.376769359963  -0.837455968870  hit
  2  -0.262069632180  -0.252632193937  -0.505264387873  hit
  3  -0.212344305472  -0.221954622675  -0.443909245350  stand
  4  -0.154588387501  -0.191774735846  -0.383549471693  stand
  5  -0.123356199186  -0.177735782756  -0.355471565512  stand
  6  -0.143472517226  -0.178845823282  -0.357691646564  stand
  7  -0.493091449153  -0.258201279255  -0.585150511183  hit
  8  -0.535705797940  -0.320983394567  -0.711202794432  hit
  9  -0.548658058239  -0.389558921286  -0.826051145985  hit
 10  -0.526434319555  -0.384545290572  -0.790615290656  hit
0001000100 hard 12
  1  -0.648266296496  -0.374914791124  -0.846660492508  hit
  2  -0.281306296010  -0.259825953079  -0.519651906157  hit
  3  -0.220208769391  -0.229754546363  -0.459509092725  stand
  4  -0.182857436537  -0.214040280339  -0.428080560679  stand
  5  -0.130748726417  -0.186620646361  -0.373241292723  stand
  6  -0.147292650354  -0.184087599261  -0.368175198521  stand
  7  -0.483610918230  -0.245545247897  -0.563593027921  hit
  8  -0.541216711413  -0.319225672243  -0.719326534165  hit
  9  -0.529140405356  -0.393813323180  -0.833734426639  hit
 10  -0.531573593889  -0.385856189477  -0.805908141365  hit
0010000010 hard 12
  1  -0.656577200037  -0.374483443538  -0.858216842421  hit
  2  -0.288029991127  -0.266254887194  -0.532509774389  hit
  3  -0.261815171261  -0.255711843192  -0.511423686384  hit
  4  -0.191249133500  -0.225500142483  -0.451000284965  stand
  5  -0.136833950146  -0.196611990058  -0.393223980117  stand
  6  -0.153745176030  -0.193793934242  -0.387587868484  stand
  7  -0.491583303743  -0.247103664661  -0.578757474110  hit
  8  -0.511343115632  -0.316143134265  -0.703700342158  hit
  9  -0.533416581950  -0.392171414811  -0.840364238099  hit
 10  -0.535297854690  -0.388542873960  -0.821116685479  hit
0100000001 hard 12
  1  -0.672787640232  -0.348923061088  -0.810930058925  hit
  2  -0.310992805862  -0.243407516084  -0.486824876360  hit
  3  -0.267866610675  -0.219297754477  -0.438595508955  hit
  4  -0.211838882410  -0.193955439350  -0.387910878699  hit
  5  -0.144042233531  -0.163570316909  -0.327140633817  stand
  6  -0.160378641754  -0.159435817659  -0.318871635318  hit
  7  -0.472816637384  -0.212020253670  -0.495827782945  hit
  8  -0.517356897020  -0.274473348691  -0.625855045174  hit
  9  -0.527482035900  -0.344350874558  -0.745530983208  hit
 10  -0.539571259932  -0.346617831858  -0.739113546284  hit
0000110000 hard 11
  1  -0.634013096095   0.172946404070   0.240259127183  double
  2  -0.258788223742   0.284031032770   0.567187460448  double
  3  -0.208559035789   0.314733465673   0.629466931347  double
  4  -0.149062998823   0.351969380428   0.703938760857  double
  5  -0.095545113591   0.393653480318   0.787306960636  double
  6  -0.138999158870   0.380695852023   0.761391704047  double
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  7  -0.488937522075   0.297369348042   0.500525274306  double
  8  -0.533297289895   0.229713896001   0.365672201920  double
  9  -0.544507917820   0.151959775249   0.239910950569  double
 10  -0.549500182211   0.113831692224   0.170737344417  double
0001001000 hard 11
  1  -0.641782289250   0.171306444478   0.222041253380  double
  2  -0.278423479057   0.272341228147   0.541764755199  double
  3  -0.215710381484   0.305578227660   0.611156455320  double
  4  -0.157222960410   0.342543061741   0.685086123482  double
  5  -0.124742218331   0.361679351109   0.723358702217  double
  6  -0.144697316816   0.365740796144   0.731481592287  double
  7  -0.482794975000   0.293770393125   0.487454415049  double
  8  -0.535470688058   0.221712089637   0.340213339809  double
  9  -0.549545369685   0.149504671738   0.224729221864  double
 10  -0.528223643569   0.109570813395   0.170990526979  double
0010000100 hard 11
  1  -0.650336294312   0.170919847343   0.205671902828  double
  2  -0.283577202618   0.268218033287   0.533572331288  double
  3  -0.235977701817   0.295197875793   0.590395751587  double
  4  -0.186681257797   0.312585308104   0.625170616207  double
  5  -0.132653670123   0.349535626347   0.699071252694  double
  6  -0.149704867338   0.356117191569   0.712234383137  double
  7  -0.485834972959   0.291794377603   0.472586714545  double
  8  -0.531844887108   0.220266056556   0.329975214233  double
  9  -0.529562126199   0.140061368871   0.215208484034  double
 10  -0.532204868087   0.109187162094   0.162257164459  double
0100000010 hard 11
  1  -0.657160366449   0.169644109708   0.188935205346  double
  2  -0.286593371544   0.263585350422   0.524828347443  double
  3  -0.264051481338   0.269338855412   0.538044317709  double
  4  -0.207032454751   0.300048551834   0.600097103668  double
  5  -0.138631626488   0.339824280448   0.679648560895  double
  6  -0.156394393105   0.345815751798   0.691631503597  double
  7  -0.493342007569   0.288888934407   0.455400803937  double
  8  -0.513557504923   0.215256317988   0.327730597075  double
  9  -0.523863567236   0.142600905808   0.213840029861  double
 10  -0.536793482154   0.103847414985   0.142776190603  double
0000200000 hard 10
  1  -0.653286915609   0.090561274738   0.055148869271  -0.715392590296  hit
  2  -0.255915497574   0.223862221010   0.446441680917  -0.270358993081  double
  3  -0.206171838286   0.254781718869   0.509563437738  -0.177264884419  double
  4  -0.146290657592   0.294911075872   0.589822151743  -0.065368771196  double
  5  -0.090048390640   0.347345763138   0.694691526276   0.057221091691  double
  6  -0.111669289731   0.361822562204   0.723645124407   0.030293363671  double
  7  -0.484469352219   0.279058637411   0.466340109353  -0.352238261412  double
  8  -0.530852624438   0.207838473033   0.322884659022  -0.508691089894  double
  9  -0.540668107570   0.120345233132   0.174552624695  -0.671853011991  double
 10  -0.545138744833   0.038319161507   0.018005473345  -0.735443034246  hit
0001010000 hard 10
  1  -0.636118371724   0.081506363506   0.048615306941  hit
  2  -0.275236262212   0.217493721192   0.432186766778  double
  3  -0.212067508068   0.250439149428   0.500878298855  double
  4  -0.151620782188   0.292761353889   0.585522707778  double
  5  -0.097978522181   0.338824533253   0.677649066505  double
  6  -0.139190053742   0.331647548483   0.663295096966  double
  7  -0.478620495592   0.285588098293   0.475371190785  double
  8  -0.533076549004   0.207506449518   0.317152695008  double
  9  -0.545472318381   0.117491536847   0.164632708249  double
 10  -0.550442382505   0.035322094058   0.011218550798  hit
0010001000 hard 10
  1  -0.644758910669   0.086842688099   0.040178659449  hit
  2  -0.280804755065   0.214421304874   0.425979592567  double
  3  -0.231628396068   0.246306384174   0.492141228594  double
  4  -0.160912203100   0.285178693849   0.570357387698  double
  5  -0.127734089955   0.307505499259   0.615010998518  double
  6  -0.146112447834   0.318049603424   0.636099206848  double
  7  -0.485024378769   0.277201339257   0.444697064776  double
  8  -0.526077048058   0.217139056959   0.326863504900  double
  9  -0.550044179643   0.117568732224   0.153704735609  double
 10  -0.528007794047   0.030281200389   0.014483932711  hit
0100000100 hard 10
  1  -0.651889077725   0.086447622780   0.025083490586  hit
  2  -0.282091984478   0.213044833615   0.424480339800  double
  3  -0.238251785461   0.241477229708   0.482478296720  double
  4  -0.202467987751   0.252464493360   0.504928986721  double
  5  -0.135541754596   0.295987052742   0.591974105483  double
  6  -0.151322663519   0.307533477329   0.615066954658  double
  7  -0.487637570382   0.267639030131   0.416637901188  double
  8  -0.534063996141   0.207287667520   0.294474042083  double
  9  -0.520050015914   0.120803495385   0.174427797338  double
 10  -0.532853371831   0.031007763702   0.005949895685  hit
0001100000 hard  9
  1  -0.654532697680  -0.070214846928  -0.395770995763  hit
  2  -0.273342090996   0.093337553102   0.113509213419  double
  3  -0.209737265768   0.141082419507   0.195604685789  double
  4  -0.148879197168   0.189607796358   0.297004323271  double
  5  -0.092476837200   0.246436339282   0.414994705208  double
  6  -0.112880631792   0.263315546403   0.443310146302  double
  7  -0.473057226468   0.201986312293   0.190459118956  hit
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  8  -0.530617624206   0.108485452793   0.000738542554  hit
  9  -0.541653170112  -0.050632799265  -0.278961155729  hit
 10  -0.546164886609  -0.137521558967  -0.438369494774  hit
0010010000 hard  9
  1  -0.638344006762  -0.078494633687  -0.401328835674  hit
  2  -0.278473931535   0.092149298319   0.111874257161  double
  3  -0.227965212747   0.138943886393   0.189257574743  double
  4  -0.155475361728   0.186021488296   0.289638180471  double
  5  -0.100904035145   0.234973358723   0.391603346823  double
  6  -0.141538154636   0.231326067344   0.379215124615  double
  7  -0.479852389096   0.197719913128   0.175697427853  hit
  8  -0.523698541361   0.117522141319   0.014198621612  hit
  9  -0.545955605632  -0.051069641781  -0.289688819921  hit
 10  -0.550310474464  -0.140495002064  -0.444545669074  hit
0100001000 hard  9
  1  -0.645559081203  -0.082784629409  -0.402057009311  hit
  2  -0.280228720320   0.083487577311   0.117401691709  double
  3  -0.233898210203   0.130824800157   0.194601689384  double
  4  -0.176727089966   0.167601298215   0.271815498866  double
  5  -0.130566512075   0.195514707503   0.332121485182  double
  6  -0.148738109107   0.209711280406   0.357360866561  double
  7  -0.485802506191   0.183613973872   0.160008211701  hit
  8  -0.528339840923   0.107593553396   0.007362344346  hit
  9  -0.540546438349  -0.052383579170  -0.274061563562  hit
 10  -0.528700838169  -0.154042709853  -0.435521325236  hit
0002000000 hard  8
  1  -0.655759168538  -0.208971827399  -0.796495530065  -0.577197989057  hit
  2  -0.290133266776  -0.012615619383  -0.184577326862  -0.255762416824  hit
  3  -0.213730377909   0.028835931491  -0.082465369305  -0.132395858607  hit
  4  -0.151693527177   0.097855255896   0.044128304904  -0.017427876594  hit
  5  -0.094896587995   0.153926501483   0.162314209377   0.090804010400  double
  6  -0.114033553938   0.175289720432   0.193183623463   0.076505621308  double
  7  -0.462716482092   0.111348886533  -0.108352780510  -0.253438837453  hit
  8  -0.529260898689  -0.054359103677  -0.447061808344  -0.368983019856  hit
  9  -0.542638232654  -0.204340895034  -0.701087440913  -0.517074994733  hit
 10  -0.547191028384  -0.241025239706  -0.738996075536  -0.599050750245  hit
0010100000 hard  8
  1  -0.656760894123  -0.217133357312  -0.793964167250  hit
  2  -0.275743649537  -0.016523215938  -0.167800157238  hit
  3  -0.226529007842   0.019485248012  -0.083155648450  hit
  4  -0.152839136224   0.086611281421   0.041092217785  hit
  5  -0.095494735721   0.141067279173   0.154983213542  double
  6  -0.115243513893   0.163695752804   0.189928883965  double
  7  -0.475347110562   0.093111516353  -0.132299136251  hit
  8  -0.520171472264  -0.056499986683  -0.428184577036  hit
  9  -0.542146635797  -0.216991296247  -0.702933413396  hit
 10  -0.546016076065  -0.250837807746  -0.738023413360  hit
0100010000 hard  8
  1  -0.639160441411  -0.225772272922  -0.799095150282  hit
  2  -0.277108047051  -0.012996524110  -0.162783294738  hit
  3  -0.231141804560   0.023799685042  -0.078252453446  hit
  4  -0.171326769819   0.081062049609   0.023309236406  hit
  5  -0.103818941813   0.130629787231   0.130582861892  hit
  6  -0.144181266647   0.132138979608   0.124194276351  hit
  7  -0.481633433698   0.091844928350  -0.148120087793  hit
  8  -0.524927419973  -0.055928169889  -0.437134663187  hit
  9  -0.536502811536  -0.208316621378  -0.691585550859  hit
 10  -0.551019164821  -0.250052981568  -0.745150533534  hit
0011000000 hard  7
  1  -0.657966569561  -0.345047909329  -1.126423325632  hit
  2  -0.292521203876  -0.122957273928  -0.429083010108  hit
  3  -0.229953470344  -0.081489688830  -0.323683361864  hit
  4  -0.155934130142  -0.016568645494  -0.177708453375  hit
  5  -0.097993129068   0.048985648294  -0.061355296509  hit
  6  -0.116457654427   0.059191643404  -0.056865314868  hit
  7  -0.464878513626  -0.070035596222  -0.542576764698  hit
  8  -0.519886128120  -0.227965865215  -0.848583128874  hit
  9  -0.542009973053  -0.303648258861  -0.946163734047  hit
 10  -0.547042217841  -0.334778624438  -0.955482603816  hit
0100100000 hard  7
  1  -0.657528880741  -0.331114688560  -1.125236863549  hit
  2  -0.274153433312  -0.099368504697  -0.401783371562  hit
  3  -0.228858222922  -0.061451285786  -0.310675564879  hit
  4  -0.169574202181  -0.010319895941  -0.194640934615  hit
  5  -0.098372783516   0.056768207398  -0.070318319650  hit
  6  -0.117899804973   0.069596420374  -0.061037814848  hit
  7  -0.477078095745  -0.066929360374  -0.567246654361  hit
  8  -0.522459065892  -0.217531982620  -0.853460343790  hit
  9  -0.531594566176  -0.283812351426  -0.926277416117  hit
 10  -0.546773708926  -0.322020532727  -0.955173120868  hit
0020000000 hard  6
  1  -0.660146575389  -0.333963378117  -1.298179991019  -0.481366181738  hit
  2  -0.294866213009  -0.152948296150  -0.567470023287  -0.200181794108  hit
  3  -0.246185580999  -0.118135707772  -0.472051893808  -0.127129434386  hit
  4  -0.159690694974  -0.047439235867  -0.302167328652   0.015778576780  split
  5  -0.101455009714   0.008318224780  -0.183981844960   0.127911037531  split
  6  -0.118892589145   0.013895770139  -0.214511452790   0.122171135765  split
  7  -0.467213097011  -0.164040140625  -0.871217476905  -0.108231643613  split
  8  -0.510419479913  -0.230703166089  -0.999660447788  -0.265461958658  hit
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  9  -0.542453094825  -0.309365918121  -1.065288273244  -0.424629803881  hit
 10  -0.545671973097  -0.343585423006  -1.071266722109  -0.516912110915  hit
0101000000 hard  6
  1  -0.658711576620  -0.334901050097  -1.295655190320  hit
  2  -0.290966669762  -0.150708443391  -0.560133584537  hit
  3  -0.232274492712  -0.106137533822  -0.445515804346  hit
  4  -0.172105622158  -0.055246914415  -0.323648502928  hit
  5  -0.101371304956   0.008890106651  -0.185536310001  hit
  6  -0.119123349512   0.014112779737  -0.215035682039  hit
  7  -0.466637707609  -0.163473202135  -0.870341765358  hit
  8  -0.522155209872  -0.233909929042  -1.023132273245  hit
  9  -0.532529284806  -0.303726453470  -1.045309072526  hit
 10  -0.546578416501  -0.345110867951  -1.073121761349  hit
0110000000 hard  5
  1  -0.659375410584  -0.291565494674  -1.318750821169  hit
  2  -0.293289487154  -0.131386685571  -0.586578974309  hit
  3  -0.248451595834  -0.098249973662  -0.496903191669  hit
  4  -0.175850171772  -0.041018362587  -0.351700343544  hit
  5  -0.104257093388   0.021504824279  -0.208514186776  hit
  6  -0.121914642138   0.019233082934  -0.243829284276  hit
  7  -0.468910126551  -0.118961541724  -0.937820253103  hit
  8  -0.512722486596  -0.180628832712  -1.025444973193  hit
  9  -0.532953684936  -0.262095856472  -1.065907369872  hit
 10  -0.546374787032  -0.307756563306  -1.092749574063  hit
0200000000 hard  4
  1  -0.660048961161  -0.258900839322  -1.320097922322  -0.442274135143  hit
  2  -0.290802723324  -0.113174087775  -0.581605446649  -0.129314459707  hit
  3  -0.250721643406  -0.081767308530  -0.501443286812  -0.068618813840  split
  4  -0.191999070094  -0.034841990605  -0.383998140188   0.006321571102  split
  5  -0.107013988612   0.035944347428  -0.214027977223   0.136489277654  split
  6  -0.124501638919   0.032054754479  -0.249003277837   0.126811992508  split
  7  -0.471026909692  -0.091472080557  -0.942053819383  -0.050855651039  split
  8  -0.514916576379  -0.140878082084  -1.029833152758  -0.218346173534  hit
  9  -0.523454980624  -0.221986392805  -1.046909961249  -0.395947736447  hit
 10  -0.547136873930  -0.275101419051  -1.094273747860  -0.474890297756  hit
1000000001 soft 21
  1   1.500000000000   0.152525682001   0.145193769183  stand
  2   1.500000000000   0.247095034174   0.486978697352  stand
  3   1.500000000000   0.270668434238   0.536350647015  stand
  4   1.500000000000   0.298209384570   0.595070858318  stand
  5   1.500000000000   0.331495643753   0.662991287507  stand
  6   1.500000000000   0.341117159102   0.682234318203  stand
  7   1.500000000000   0.285776479909   0.467592804170  stand
  8   1.500000000000   0.221207058384   0.332681345860  stand
  9   1.500000000000   0.148863696026   0.216489533440  stand
 10   1.500000000000   0.104681908509   0.139082556247  stand
1000000010 soft 20
  1   0.680745065844   0.067504918838  -0.040656542852  stand
  2   0.655984992073   0.190762381691   0.379767998333  stand
  3   0.644125740348   0.196433865981   0.392234338848  stand
  4   0.653882758988   0.229578281966   0.459156563932  stand
  5   0.682073972877   0.268101320126   0.536202640252  stand
  6   0.694186797657   0.279876064321   0.559752128642  stand
  7   0.773193791314   0.242710948262   0.351278726887  stand
  8   0.784813844805   0.171549239433   0.229784478698  stand
  9   0.765634692752   0.096532616017   0.110516166877  stand
 10   0.554550939147   0.008646820031  -0.052039423516  stand
1000000100 soft 19
  1   0.289743228522  -0.017242144878  -0.192057344153  stand
  2   0.401625257923   0.120029027880   0.237194320473  stand
  3   0.419871896196   0.173024172056   0.346048344111  stand
  4   0.415489642766   0.186537279434   0.373074558868  stand
  5   0.460791633862   0.226529289550   0.453058579101  stand
  6   0.482353899668   0.241312965659   0.482625931318  double
  7   0.614504192650   0.221936826352   0.325331009621  stand
  8   0.607839996369   0.157746648533   0.190196697504  stand
  9   0.287945731933   0.005019781291  -0.060207420376  stand
 10   0.064313102468  -0.085536069684  -0.224373089785  stand
1000001000 soft 18
  1  -0.101014601648  -0.108385604522  -0.357778420089  stand
  2   0.135802367184   0.065247954485   0.127578207873  stand
  3   0.166816489728   0.094467202209   0.188934404418  double
  4   0.203973714893   0.156341801234   0.312683602467  double
  5   0.222295248248   0.174546797429   0.349093594857  double
  6   0.262173963689   0.192429924479   0.384859848958  double
  7   0.411952469546   0.174667380912   0.240223962657  stand
  8   0.120931058208   0.047499925559  -0.015314487046  stand
  9  -0.178831862967  -0.086957516844  -0.254496644300  hit
 10  -0.186187306439  -0.138665402505  -0.321340592914  hit
1000010000 soft 17
  1  -0.482814796225  -0.199899491315  -0.527212930766  hit
  2  -0.131767396493   0.007097780732   0.013320956371  double
  3  -0.093230740952   0.036948143291   0.073896286582  double
  4  -0.036666762715   0.077274148107   0.154548296213  double
  5   0.004662167008   0.140016510261   0.280033020522  double
  6   0.010434529426   0.133243064309   0.266486128618  double
  7  -0.089639202708   0.059646040643   0.014176718345  hit
  8  -0.385254211011  -0.064895918784  -0.229743504643  hit
  9  -0.407069558123  -0.134673789176  -0.345244185446  hit
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 10  -0.417782531607  -0.188464037066  -0.432804338482  hit
1000100000 soft 16
  1  -0.659151196762  -0.206061450252  -0.656069537080  hit
  2  -0.266381745794  -0.031723821203  -0.081846742507  hit
  3  -0.220510793360  -0.001897105265  -0.019208248379  hit
  4  -0.163591680785   0.037976111860   0.062591729069  double
  5  -0.110736472238   0.082112761519   0.148255970530  double
  6  -0.107595619850   0.115917861500   0.216665384731  double
  7  -0.467985809203  -0.023767777621  -0.189067335657  hit
  8  -0.514868683436  -0.084296355358  -0.332549699712  hit
  9  -0.525481528075  -0.166412396037  -0.452038406231  hit
 10  -0.537311243392  -0.223336992245  -0.536694083578  hit
1001000000 soft 15
  1  -0.658817601698  -0.153931072878  -0.619882188728  hit
  2  -0.283156544580  -0.011696877992  -0.069979551782  hit
  3  -0.223952986758   0.023368960448   0.002627467694  hit
  4  -0.166174982032   0.061444650792   0.084883061076  double
  5  -0.113096934709   0.107857342763   0.174984824310  double
  6  -0.109194344166   0.120256844312   0.200721278158  double
  7  -0.457616803585   0.033764353065  -0.140937310685  hit
  8  -0.514495666514  -0.035471180091  -0.314134260634  hit
  9  -0.526348439749  -0.113294558968  -0.421784353033  hit
 10  -0.538314239812  -0.170423771278  -0.496264011632  hit
1010000000 soft 14
  1  -0.661046517170  -0.100568125183  -0.595803197623  hit
  2  -0.284211460313   0.016913587018  -0.046786269606  hit
  3  -0.240149344995   0.044160358621   0.010855222231  hit
  4  -0.169927115892   0.090754534148   0.109144347455  double
  5  -0.116020000152   0.136564281519   0.203578257782  double
  6  -0.111568419383   0.147163703442   0.221859690796  double
  7  -0.460210151495   0.060464833187  -0.174539483653  hit
  8  -0.505091900523   0.035047232080  -0.254214068362  hit
  9  -0.526764039229  -0.059660056279  -0.393525409504  hit
 10  -0.538042807921  -0.123446217514  -0.479654567637  hit
1100000000 soft 13
  1  -0.661732295805  -0.067807500543  -0.593280862557  hit
  2  -0.282713434576   0.039266360130  -0.042020177179  hit
  3  -0.241148978597   0.070663854603   0.028414276386  hit
  4  -0.186079424489   0.110212986461   0.115097236657  double
  5  -0.118845850033   0.158730901179   0.212294599743  double
  6  -0.114167416071   0.168495342857   0.230210514725  double
  7  -0.461877054515   0.107386943449  -0.157238475224  hit
  8  -0.507685226980   0.039128601408  -0.312387153880  hit
  9  -0.517197957029  -0.013718071276  -0.372682106229  hit
 10  -0.538704712938  -0.088816559144  -0.478996994439  hit
2000000000 soft 12
  1  -0.663141912699  -0.030635182696  -0.598755174653   0.223931403109  split
  2  -0.274327339258   0.094776960733  -0.019359549009   0.565703802490  split
  3  -0.232311017919   0.120586135401   0.054881887601   0.612855953983  split
  4  -0.178248976452   0.145730441810   0.136650090893   0.668581932267  split
  5  -0.130085645599   0.182013722116   0.215726897142   0.732160357609  split
  6  -0.103505472999   0.199606786387   0.247914348001   0.758276018852  split
  7  -0.452479450372   0.158489177933  -0.136974870377   0.540711617521  split
  8  -0.499700231245   0.093059733984  -0.295648254875   0.406467781517  split
  9  -0.510995949413  -0.002487619086  -0.420607638303   0.289769656262  split
 10  -0.530674250923  -0.046841719533  -0.468319186909   0.194251441664  split
```

21.12 The smallest difference occurs with $\{A, 2, 3, T\}$ vs. T (0.000080304178), the largest with $\{A, A, A, A, 2, 2, 2, 2, 4\}$ vs. 7 (0.149620063291).

```
0000001010 hard 16
  7  -0.507326854118  -0.374945053600  hit
  8  -0.525806002725  -0.427790260794  hit
  9  -0.539835554663  -0.482037461884  hit
 10  -0.518034139317  -0.512008735213  hit
0000210000 hard 16
  7  -0.492695643502  -0.435797819650  hit
  8  -0.538871610000  -0.491687225616  hit
  9  -0.549580030019  -0.551862580533  stand
 10  -0.556030970533  -0.585888110334  stand
0001101000 hard 16
  7  -0.485011854065  -0.431402354172  hit
  8  -0.541018674953  -0.489936162433  hit
  9  -0.554851389860  -0.549046723723  hit
 10  -0.532972533162  -0.579650519341  stand
0004000000 hard 16
  7  -0.441721695813  -0.480076379980  stand
  8  -0.535787245571  -0.552612743644  stand
  9  -0.557291751746  -0.610287120347  stand
 10  -0.564525651830  -0.628350707176  stand
0010100100 hard 16
  7  -0.488041662984  -0.429650247208  hit
  8  -0.536114840228  -0.483944448092  hit
  9  -0.533243629636  -0.541004781049  stand
 10  -0.537241103993  -0.566471309121  stand
0012100000 hard 16
  7  -0.455198959553  -0.483715822736  stand
  8  -0.528383629627  -0.548100645185  stand
0000010001 hard 16
  7  -0.483324420261  -0.376193509207  hit
  8  -0.527006523861  -0.424822549943  hit
  9  -0.539231513534  -0.479306375180  hit
 10  -0.542951853825  -0.506929242579  hit
0001020000 hard 16
  7  -0.486460845637  -0.390989595783  hit
  8  -0.541095624371  -0.449811216628  hit
  9  -0.554623108655  -0.509296323553  hit
 10  -0.559716321144  -0.536799868202  hit
0002000100 hard 16
  7  -0.474960867683  -0.425619617889  hit
  8  -0.545526915937  -0.488918090775  hit
  9  -0.534561768391  -0.547624903743  stand
 10  -0.537494160162  -0.568289844258  stand
0010011000 hard 16
  7  -0.491915327762  -0.390948592661  hit
  8  -0.533816727796  -0.442870552652  hit
  9  -0.557755753506  -0.499429036866  hit
 10  -0.537389444608  -0.529143934786  hit
0011000010 hard 16
  7  -0.483033874756  -0.425158268998  hit
  8  -0.514402737161  -0.483833958285  hit
  9  -0.537079397116  -0.535057260948  hit
 10  -0.541488013420  -0.555709125895  stand
0020000001 hard 16
  7  -0.463981933141  -0.425669870639  hit
  8  -0.508464818146  -0.473965665720  hit
```

```
  9  -0.554157964747  -0.604176210570  stand
 10  -0.563221049858  -0.626237729870  stand
0020200000 hard 16
  7  -0.470386249696  -0.487713947688  stand
  8  -0.517180731790  -0.543363571435  stand
  9  -0.554780520080  -0.597614434194  stand
 10  -0.560523345987  -0.624202248532  stand
0030001000 hard 16
  7  -0.472532512966  -0.442132788105  hit
  8  -0.513451548250  -0.493875274213  hit
  9  -0.561991410908  -0.544831236594  hit
 10  -0.538900746223  -0.565897237402  stand
0100002000 hard 16
  7  -0.497082806018  -0.389914137630  hit
  8  -0.537030384903  -0.433129147736  hit
  9  -0.552227728743  -0.486111065710  hit
 10  -0.515857161322  -0.526451732396  stand
0100100010 hard 16
  7  -0.495757730044  -0.428227216067  hit
  8  -0.516257609300  -0.476047702961  hit
  9  -0.526282401060  -0.525187175677  hit
 10  -0.542158675386  -0.559519262054  stand
0101200000 hard 16
  7  -0.469552993997  -0.486526436283  stand
  8  -0.532375490957  -0.540550602872  stand
  9  -0.541511447699  -0.594275284592  stand
 10  -0.561560307108  -0.629635614732  stand
0110110000 hard 16
  7  -0.476654918929  -0.445363141240  hit
  8  -0.523292346789  -0.492861220515  hit
  9  -0.547082998489  -0.543828497053  hit
 10  -0.565869342797  -0.577982101443  stand
0120000100 hard 16
  7  -0.474763113624  -0.439248302632  hit
  8  -0.522122212718  -0.485393983246  hit
  9  -0.529170300713  -0.532951295176  stand
 10  -0.544163108873  -0.558295662362  stand
0130100000 hard 16
  7  -0.453941775395  -0.499905024348  stand
  8  -0.499813710147  -0.546919252512  stand
  9  -0.548467959080  -0.590245656859  stand
 10  -0.566818391703  -0.615619440222  stand
0200101000 hard 16
  7  -0.483129902970  -0.444338409173  hit
  8  -0.529108912145  -0.483518827174  hit
  9  -0.539826774635  -0.530657459081  hit
 10  -0.541361527356  -0.574260185437  stand
0203000000 hard 16
  7  -0.438776786323  -0.494560498173  stand
  8  -0.523581671881  -0.548501162724  stand
  9  -0.537977816517  -0.594135903436  stand
 10  -0.567975591321  -0.623644283911  stand
0211100000 hard 16
  7  -0.453427759912  -0.498557770704  stand
  8  -0.514462469062  -0.543835801024  stand
  9  -0.536021303853  -0.587073526776  stand
 10  -0.569411721328  -0.621453792676  stand
0240000000 hard 16
  7  -0.438446359842  -0.514029469131  stand
  8  -0.482498161069  -0.551686407220  stand
  9  -0.539911100853  -0.582225722430  stand
 10  -0.568583315471  -0.605456707287  stand
0300200000 hard 16
  7  -0.469692516389  -0.502103762711  stand
  8  -0.517132095040  -0.535683652405  stand
  9  -0.522829717680  -0.576307690654  stand
 10  -0.572214549834  -0.624986489035  stand
0310001000 hard 16
  7  -0.468876427882  -0.455470500164  hit
  8  -0.514359453300  -0.485323788506  hit
  9  -0.536434012443  -0.521868789714  hit
 10  -0.547457574733  -0.565012125959  stand
0400000100 hard 16
  7  -0.476154804173  -0.453072645046  hit
  8  -0.522610340465  -0.476387787572  hit
  9  -0.502040795991  -0.509073425389  stand
 10  -0.553436596626  -0.558418048232  stand
0410100000 hard 16
  7  -0.453815875308  -0.515966710544  stand
  8  -0.499009829271  -0.539568370119  stand
  9  -0.517865610157  -0.568336020191  stand
 10  -0.577789212710  -0.615820868947  stand
1000010010 hard 16
  7  -0.489501600284  -0.378144415520  hit
  8  -0.511229912011  -0.424722047033  hit
  9  -0.524185403287  -0.478354248441  hit
 10  -0.537348196738  -0.511685779769  hit
1000300000 hard 16
  7  -0.473960930486  -0.481468451457  stand
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  9  -0.540450145381  -0.523394877812  hit
 10  -0.543884019567  -0.544348388090  stand
0021010000 hard 16
  7  -0.463409941560  -0.442278958234  hit
  8  -0.521648087015  -0.500395279745  hit
  9  -0.557245378759  -0.554215916481  hit
 10  -0.566207700118  -0.574529624446  stand
0041000000 hard 16
  7  -0.439970502971  -0.497602199741  stand
  8  -0.496421351439  -0.554887101316  stand
  9  -0.556485867828  -0.600179359361  stand
 10  -0.567337662533  -0.611384284774  stand
0100010100 hard 16
  7  -0.494628399732  -0.388199762322  hit
  8  -0.539412150627  -0.433969099340  hit
  9  -0.527492618676  -0.487927581752  hit
 10  -0.542529230598  -0.521460416416  hit
0101000001 hard 16
  7  -0.463427576542  -0.424416236576  hit
  8  -0.519852694676  -0.471705059221  hit
  9  -0.530971288080  -0.520009672331  hit
 10  -0.544807737703  -0.549086768759  stand
0102010000 hard 16
  7  -0.464014837908  -0.440942967522  hit
  8  -0.533236198470  -0.497947461866  hit
  9  -0.547643329953  -0.550585973169  stand
 10  -0.565821981323  -0.580170247020  stand
0111001000 hard 16
  7  -0.470599035514  -0.440820397690  hit
  8  -0.526375045272  -0.491317568138  hit
  9  -0.551125963629  -0.541252315773  hit
 10  -0.541223324688  -0.571482984390  stand
0122000000 hard 16
  7  -0.439067115884  -0.495804531121  stand
  8  -0.510131684691  -0.551866134634  stand
  9  -0.546564260503  -0.597190935365  stand
 10  -0.569770552130  -0.617585898497  stand
0200020000 hard 16
  7  -0.483551781927  -0.403331167398  hit
  8  -0.526992940803  -0.442574468037  hit
  9  -0.541981463906  -0.489523308797  hit
 10  -0.569327137592  -0.530924737398  hit
0201000100 hard 16
  7  -0.474552489806  -0.438066360273  hit
  8  -0.534808803545  -0.482748670774  hit
  9  -0.518397804296  -0.529328372269  stand
 10  -0.544298888838  -0.563886063043  stand
0210000010 hard 16
  7  -0.482929897667  -0.437936970449  hit
  8  -0.502036369774  -0.477330754117  hit
  9  -0.522973922195  -0.517242784692  hit
 10  -0.548278392768  -0.551912252398  stand
0220010000 hard 16
  7  -0.460786959088  -0.456533272778  hit
  8  -0.507368723576  -0.494970729297  hit
  9  -0.542873299178  -0.535253736621  hit
 10  -0.572178368589  -0.568642872142  hit
0300000001 hard 16
  7  -0.463053324659  -0.437369905074  hit
  8  -0.508328179313  -0.465915736576  hit
  9  -0.516483908111  -0.502599899717  hit
 10  -0.551273152637  -0.543783325427  hit
0301010000 hard 16
  7  -0.462118637430  -0.455486460694  hit
  8  -0.519330101998  -0.492143066748  hit
  9  -0.531631763142  -0.531718245721  stand
 10  -0.573538028055  -0.574411303507  stand
0321000000 hard 16
  7  -0.437976721709  -0.512453561637  stand
  8  -0.496391293209  -0.548241703781  stand
  9  -0.528566007426  -0.579320599780  stand
 10  -0.573056213693  -0.611852690148  stand
0402000000 hard 16
  7  -0.438312929691  -0.511441721028  stand
  8  -0.510431462615  -0.544451843612  stand
  9  -0.518054673630  -0.576375158213  stand
 10  -0.573240353438  -0.618037370310  stand
1000001100 hard 16
  7  -0.487508403901  -0.378460334081  hit
  8  -0.534313212585  -0.422406866550  hit
  9  -0.525051137409  -0.480988304909  hit
 10  -0.511572586799  -0.517238264798  stand
1000100001 hard 16
  7  -0.464177879939  -0.419715924119  hit
  8  -0.512474461175  -0.462138740524  hit
  9  -0.523799658917  -0.515155370707  hit
 10  -0.536043755470  -0.549644636262  stand
1001110000 hard 16
  7  -0.466121679694  -0.435976617955  hit
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  8  -0.524978943876  -0.530581970417  stand
  9  -0.534572946171  -0.589508210080  stand
 10  -0.548862092754  -0.630845364535  stand
1002001000 hard 16
  7  -0.459312288659  -0.431641610411  hit
  8  -0.527852324419  -0.486028613570  hit
  9  -0.545363128148  -0.543625465563  hit
 10  -0.532567795800  -0.575223606198  stand
1010101000 hard 16
  7  -0.473337279843  -0.435920899375  hit
  8  -0.518019205395  -0.481332737548  hit
  9  -0.544958396713  -0.537074030088  hit
 10  -0.530244257076  -0.572753536474  stand
1013000000 hard 16
  7  -0.427864743623  -0.486134632731  stand
  8  -0.513114847207  -0.545673926495  stand
  9  -0.542844281856  -0.600492226870  stand
 10  -0.562199527532  -0.621705445391  stand
1021100000 hard 16
  7  -0.442649577905  -0.490046120452  stand
  8  -0.504140525188  -0.541363875239  stand
  9  -0.542241329029  -0.593892037058  stand
 10  -0.559235560425  -0.619708874949  stand
1100011000 hard 16
  7  -0.478693262472  -0.393988239045  hit
  8  -0.523534818653  -0.431447484091  hit
  9  -0.536941240508  -0.482055970902  hit
 10  -0.536235922976  -0.526188019016  hit
1101000010 hard 16
  7  -0.470914106069  -0.428520044611  hit
  8  -0.504817112768  -0.473086611890  hit
  9  -0.515286517726  -0.520105469262  stand
 10  -0.539474299944  -0.554742882384  stand
1110000001 hard 16
  7  -0.451299116413  -0.429255184942  hit
  8  -0.498591813997  -0.464198078814  hit
  9  -0.519576779594  -0.508582834368  hit
 10  -0.542177311306  -0.542257615484  stand
1111010000 hard 16
  7  -0.450375887125  -0.446601580345  hit
  8  -0.510814993151  -0.490185458683  hit
  9  -0.535301844023  -0.538952009748  stand
 10  -0.563848014254  -0.572579881348  stand
1131000000 hard 16
  7  -0.426736737612  -0.503456388871  stand
  8  -0.486141814793  -0.546374955479  stand
  9  -0.533832236963  -0.587301691974  stand
 10  -0.564092485857  -0.610424010075  stand
1201001000 hard 16
  7  -0.456938390717  -0.445678392259  hit
  8  -0.517199618853  -0.480977050836  hit
  9  -0.529412363294  -0.525399821789  hit
 10  -0.538292450894  -0.569404065663  stand
1212000000 hard 16
  7  -0.425769460262  -0.502163103488  stand
  8  -0.500516337510  -0.542997741581  stand
  9  -0.523584913228  -0.583784793178  stand
 10  -0.565312155904  -0.616572094298  stand
1300000010 hard 16
  7  -0.472129245087  -0.443054455307  hit
  8  -0.492377287672  -0.467469421405  hit
  9  -0.499736495389  -0.501632838213  stand
 10  -0.547890556652  -0.550432798597  stand
1310010000 hard 16
  7  -0.448910460246  -0.462760773816  stand
  8  -0.496339478571  -0.485864396417  hit
  9  -0.519709942796  -0.519396941213  hit
 10  -0.571501007973  -0.566103099828  hit
1400001000 hard 16
  7  -0.456416772866  -0.462310320297  stand
  8  -0.505154371689  -0.476151559623  hit
  9  -0.513217009247  -0.505293900024  hit
 10  -0.546350092659  -0.562229744164  stand
2000002000 hard 16
  7  -0.473971371370  -0.385147896526  hit
  8  -0.515485259099  -0.418964921369  hit
  9  -0.535123452323  -0.474639902332  hit
 10  -0.504829669575  -0.522205086479  stand
2000100010 hard 16
  7  -0.472407430121  -0.424051453370  hit
  8  -0.496121609604  -0.462976934619  hit
  9  -0.508183615298  -0.515402984144  stand
 10  -0.529762244397  -0.555632402151  stand
2001200000 hard 16
  7  -0.444733081506  -0.483947280644  stand
  8  -0.512644417135  -0.527804239981  stand
  9  -0.524109504696  -0.585294120543  stand
 10  -0.547553151570  -0.627072122938  stand
2010110000 hard 16
  8  -0.527066876907  -0.487763253035  hit
  9  -0.539510070541  -0.546293273580  stand
 10  -0.555179007652  -0.581247388630  stand
1010020000 hard 16
  7  -0.472506024868  -0.394956326004  hit
  8  -0.519310446908  -0.439992752136  hit
  9  -0.544094225638  -0.496565380459  hit
 10  -0.558273004890  -0.528863664881  hit
1011000100 hard 16
  7  -0.462540125464  -0.429965037844  hit
  8  -0.524358505719  -0.480266029012  hit
  9  -0.522373035322  -0.536252733382  stand
 10  -0.535813786361  -0.561824562182  stand
1020000010 hard 16
  7  -0.472072968687  -0.429705129828  hit
  8  -0.491658962412  -0.475490263833  hit
  9  -0.526994601108  -0.524162693730  hit
 10  -0.537466771771  -0.549131858021  stand
1030010000 hard 16
  7  -0.450505899246  -0.448036833652  hit
  8  -0.498228623115  -0.492574693536  hit
  9  -0.546584571561  -0.543014376955  hit
 10  -0.561432059918  -0.566957868999  stand
1100100100 hard 16
  7  -0.475661347369  -0.433278397337  hit
  8  -0.527633774100  -0.473136236698  hit
  9  -0.510219106310  -0.524921831551  stand
 10  -0.535497775809  -0.565291498660  stand
1102100000 hard 16
  7  -0.441387681702  -0.488922949418  stand
  8  -0.519129185337  -0.538328724067  stand
  9  -0.530999643529  -0.590132342865  stand
 10  -0.558923469416  -0.625445344843  stand
1110200000 hard 16
  7  -0.457619524480  -0.493125473198  stand
  8  -0.507580602693  -0.534022045976  stand
  9  -0.530326948064  -0.583070463943  stand
 10  -0.558857055450  -0.623249443040  stand
1120001000 hard 16
  7  -0.458934763944  -0.446558713337  hit
  8  -0.503520695411  -0.483734071923  hit
  9  -0.540691724337  -0.529468860595  hit
 10  -0.537160760038  -0.563855498697  stand
1200110000 hard 16
  7  -0.464597992125  -0.450372110185  hit
  8  -0.512856968839  -0.482847781132  hit
  9  -0.523437184669  -0.527604543940  stand
 10  -0.564953717036  -0.575849145981  stand
1210000100 hard 16
  7  -0.462723004825  -0.443929677282  hit
  8  -0.513200529087  -0.475308978694  hit
  9  -0.506411445551  -0.517158683576  stand
 10  -0.542581029331  -0.556807522863  stand
1220100000 hard 16
  7  -0.441516776222  -0.506496521430  stand
  8  -0.489773445499  -0.538536443809  stand
  9  -0.524297083140  -0.576223799834  stand
 10  -0.565232177541  -0.614417547043  stand
1301100000 hard 16
  7  -0.441115897243  -0.505652913687  stand
  8  -0.505454226243  -0.535113634239  stand
  9  -0.511048660169  -0.572529207888  stand
 10  -0.566597334546  -0.620278813389  stand
1330000000 hard 16
  7  -0.426232592327  -0.521854089900  stand
  8  -0.472234430346  -0.544353147996  stand
  9  -0.516444937177  -0.568798459057  stand
 10  -0.567113719026  -0.604031277854  stand
1411000000 hard 16
  7  -0.425981874053  -0.520813182896  stand
  8  -0.486895860804  -0.540532371297  stand
  9  -0.504179158901  -0.565391988896  stand
 10  -0.570488516367  -0.610362026364  stand
2000010100 hard 16
  7  -0.470038496285  -0.383421200042  hit
  8  -0.520241119665  -0.419481642730  hit
  9  -0.508226511398  -0.477305451924  hit
 10  -0.531429972118  -0.517055148724  hit
2001000001 hard 16
  7  -0.438800574394  -0.420720926278  hit
  8  -0.499708975203  -0.458612663682  hit
  9  -0.512904272378  -0.510067678546  hit
 10  -0.534772735784  -0.545325263809  stand
2002010000 hard 16
  7  -0.439114185937  -0.437328996227  hit
  8  -0.513332548293  -0.484078025255  hit
  9  -0.529023150762  -0.541518218013  stand
 10  -0.554603418663  -0.576234024307  stand
2011001000 hard 16
```

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 7  -0.452975924911  -0.441932383816  hit
 8  -0.503179741881  -0.479843769385  hit
 9  -0.528497308780  -0.534594617981  stand
10  -0.553106419068  -0.574144049252  stand
2020000100 hard 16
 7  -0.450628914446  -0.435806863687  hit
 8  -0.501835712731  -0.472440748917  hit
 9  -0.510937484042  -0.525054523728  stand
10  -0.532697253416  -0.555181838297  stand
2030100000 hard 16
 7  -0.430978201648  -0.497968131530  stand
 8  -0.478161219466  -0.535695348852  stand
 9  -0.531378801657  -0.583954787046  stand
10  -0.553442694528  -0.613078627942  stand
2100101000 hard 16
 7  -0.459158062803  -0.441137691462  hit
 8  -0.508931844062  -0.471221966396  hit
 9  -0.522031816066  -0.520600600908  hit
10  -0.528560598434  -0.570819680638  stand
2103000000 hard 16
 7  -0.412935814947  -0.492720928417  stand
 8  -0.503582295417  -0.536450390800  stand
 9  -0.520176401402  -0.586936970930  stand
10  -0.556485885952  -0.620946594309  stand
2111100000 hard 16
 7  -0.428688644815  -0.496790290128  stand
 8  -0.494573180963  -0.532650951494  stand
 9  -0.517983497564  -0.579757458304  stand
10  -0.556290390083  -0.618810187359  stand
2140000000 hard 16
 7  -0.415269732159  -0.512822836203  stand
 8  -0.460598070865  -0.542113356536  stand
 9  -0.522315243841  -0.577798227486  stand
10  -0.556493260946  -0.603099669555  stand
2200200000 hard 16
 7  -0.444844910767  -0.500599511620  stand
 8  -0.498699116056  -0.525466678235  stand
 9  -0.504014403277  -0.567848798331  stand
10  -0.557307756638  -0.622262087067  stand
2210001000 hard 16
 7  -0.445005861767  -0.453003623128  stand
 8  -0.493945368931  -0.474491495459  hit
 9  -0.518064200878  -0.513338031478  hit
10  -0.535668759475  -0.561511795793  stand
2300000100 hard 16
 7  -0.450569411261  -0.450718982494  stand
 8  -0.504857886594  -0.466059190069  hit
 9  -0.481946193971  -0.500568947449  stand
10  -0.541236751350  -0.555046505860  stand
2310100000 hard 16
 7  -0.429092173661  -0.515295779093  stand
 8  -0.480207082575  -0.531063860747  stand
 9  -0.498315604796  -0.561570866740  stand
10  -0.564065066374  -0.613060751811  stand
2420000000 hard 16
 7  -0.413952380260  -0.532051533564  stand
 8  -0.462138712110  -0.538073644486  stand
 9  -0.491286497583  -0.554782370211  stand
10  -0.566445709753  -0.602305665780  stand
3000100100 hard 16
 7  -0.449655971787  -0.430731477961  hit
 8  -0.507589107892  -0.459459218048  hit
 9  -0.491643723090  -0.515142905564  stand
10  -0.523189517918  -0.562310368357  stand
3002100000 hard 16
 7  -0.415006071847  -0.487849399398  stand
 8  -0.498181256151  -0.525383893223  stand
 9  -0.513210488593  -0.581981102656  stand
10  -0.546598574083  -0.623016603148  stand
3010200000 hard 16
 7  -0.432813075246  -0.492114329443  stand
 8  -0.486331592401  -0.521970138955  stand
 9  -0.512719420660  -0.574825354264  stand
10  -0.544623980670  -0.620879292591  stand
3020001000 hard 16
 7  -0.435080922943  -0.444631176265  stand
 8  -0.480644245987  -0.471062350813  hit
 9  -0.523230822664  -0.521315783227  hit
10  -0.525685730760  -0.560613385180  stand
3100110000 hard 16
 7  -0.439211006426  -0.448594029183  stand
 8  -0.492943370992  -0.470612471445  hit
 9  -0.503771283862  -0.518039172073  stand
10  -0.552126520414  -0.572107775644  stand
3110000100 hard 16
 7  -0.436823465184  -0.442170187956  stand
 8  -0.493031195187  -0.463132443332  hit
 9  -0.487126430008  -0.508998129609  stand
10  -0.531173636089  -0.553745725518  stand
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 7  -0.446839928618  -0.437299622387  hit
 8  -0.505073006464  -0.478047452811  hit
 9  -0.533792788495  -0.532359066853  hit
10  -0.529934958159  -0.568033309314  stand
2022000000 hard 16
 7  -0.414495499249  -0.493749825939  stand
 8  -0.489166745105  -0.539764850284  stand
 9  -0.528943433198  -0.591107165980  stand
10  -0.558262194480  -0.614962806104  stand
2100020000 hard 16
 7  -0.459533545427  -0.399164622601  hit
 8  -0.507821225114  -0.429409542284  hit
 9  -0.521797638380  -0.479036387202  hit
10  -0.557803800616  -0.525756387084  hit
2101000100 hard 16
 7  -0.448975491751  -0.434744001554  hit
 8  -0.515858250989  -0.469927396999  hit
 9  -0.499662521503  -0.520423849038  stand
10  -0.532865806967  -0.560661647066  stand
2110000010 hard 16
 7  -0.459740017751  -0.434439885711  hit
 8  -0.481714706633  -0.465686265203  hit
 9  -0.504265535610  -0.508991792079  stand
10  -0.536781283081  -0.547939197055  stand
2120010000 hard 16
 7  -0.437118745732  -0.453879431843  stand
 8  -0.487047240904  -0.483447999134  hit
 9  -0.523724100525  -0.527576577432  stand
10  -0.560445483891  -0.564743844735  stand
2200000001 hard 16
 7  -0.438384551302  -0.434665494349  hit
 8  -0.489117230371  -0.455244462750  hit
 9  -0.497329207137  -0.493050778498  hit
10  -0.540630100461  -0.539906506384  hit
2201010000 hard 16
 7  -0.437130317845  -0.452927801962  stand
 8  -0.500393667382  -0.480757618728  hit
 9  -0.511936560320  -0.523022523264  stand
10  -0.561613519600  -0.570397963862  stand
2221000000 hard 16
 7  -0.413196691756  -0.511462961572  stand
 8  -0.476221292415  -0.538705705651  stand
 9  -0.509911958905  -0.573830796732  stand
10  -0.561168990190  -0.609288837777  stand
2302000000 hard 16
 7  -0.412102182182  -0.510738119572  stand
 8  -0.491588889462  -0.534924079670  stand
 9  -0.499132480301  -0.569790528058  stand
10  -0.560892006642  -0.615380786461  stand
2400010000 hard 16
 7  -0.437136006365  -0.471175822665  stand
 8  -0.485591266363  -0.477725564283  hit
 9  -0.494645834408  -0.502900776667  stand
10  -0.571346254516  -0.563242283111  hit
3000011000 hard 16
 7  -0.454102305169  -0.390488109262  hit
 8  -0.502131381707  -0.417170801410  hit
 9  -0.518145196002  -0.471038627592  hit
10  -0.525255657349  -0.521667816345  hit
3001000010 hard 16
 7  -0.446049124753  -0.425618405127  hit
 8  -0.483619890951  -0.459755478354  hit
 9  -0.496719761533  -0.511144351970  stand
10  -0.528576723305  -0.550928852944  stand
3010000001 hard 16
 7  -0.426370028430  -0.427070016157  stand
 8  -0.477173219994  -0.451684312465  hit
 9  -0.501111505055  -0.499324336611  hit
10  -0.532059614230  -0.538620609938  stand
3011010000 hard 16
 7  -0.425040576742  -0.444593754809  stand
 8  -0.489734800398  -0.476887280279  hit
 9  -0.516145573718  -0.530624411197  stand
10  -0.552665357898  -0.568837967431  stand
3031000000 hard 16
 7  -0.401878113378  -0.503095169725  stand
 8  -0.463623288226  -0.535055382319  stand
 9  -0.516040520266  -0.582162231794  stand
10  -0.552446656353  -0.608107279636  stand
3101001000 hard 16
 7  -0.431444662982  -0.443867419161  stand
 8  -0.495799335221  -0.468474554583  hit
 9  -0.511261468096  -0.516141518772  stand
10  -0.527075725543  -0.566088224865  stand
3112000000 hard 16
 7  -0.399408537199  -0.502011489409  stand
 8  -0.479393184335  -0.531705226377  stand
 9  -0.505089618738  -0.577517271441  stand
10  -0.554001943481  -0.614117811116  stand
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3120100000 hard 16
  7  -0.416711417718  -0.506414837998  stand
  8  -0.468290472998  -0.528204974738  stand
  9  -0.505980956555  -0.569763369076  stand
 10  -0.552020774013  -0.612003066604  stand
3201100000 hard 16
  7  -0.414481569197  -0.505803115159  stand
  8  -0.485756765450  -0.524781012852  stand
  9  -0.492052851667  -0.564984665552  stand
 10  -0.553302189902  -0.617813094186  stand
3230000000 hard 16
  7  -0.401282432196  -0.522663309232  stand
  8  -0.450380082513  -0.535835767991  stand
  9  -0.497443481079  -0.564295444897  stand
 10  -0.555309440850  -0.601750934896  stand
3311000000 hard 16
  7  -0.399403333479  -0.521900125603  stand
  8  -0.466804418875  -0.532010065469  stand
  9  -0.484382681036  -0.559780406774  stand
 10  -0.558506166694  -0.607923425781  stand
4000020000 hard 16
  7  -0.433430979432  -0.397178068670  hit
  8  -0.486610861880  -0.414968249330  hit
  9  -0.501130910433  -0.468556783021  hit
 10  -0.546897546898  -0.520935634606  hit
4001000100 hard 16
  7  -0.421304856359  -0.433722968278  stand
  8  -0.494664496026  -0.455968443858  hit
  9  -0.480556394510  -0.511509825860  stand
 10  -0.522218866405  -0.557762360116  stand
4010000010 hard 16
  7  -0.434720431771  -0.433207956725  hit
  8  -0.459076053461  -0.452985258686  hit
  9  -0.485365280714  -0.500818846707  stand
 10  -0.525695493137  -0.544273093139  stand
4020010000 hard 16
  7  -0.411500836090  -0.453682241374  stand
  8  -0.464455756004  -0.470876023202  stand
  9  -0.504266389689  -0.520041964086  stand
 10  -0.549089877791  -0.561218088019  stand
4100000001 hard 16
  7  -0.411732919504  -0.434254286920  stand
  8  -0.467717037484  -0.443586405611  hit
  9  -0.477868384845  -0.483454388304  stand
 10  -0.530507735159  -0.536298566248  stand
4101010000 hard 16
  7  -0.410052227976  -0.452803413499  stand
  8  -0.479278237361  -0.468299941928  hit
  9  -0.491779066084  -0.514395075052  stand
 10  -0.550478449514  -0.566743676535  stand
4121000000 hard 16
  7  -0.386444808736  -0.513215318829  stand
  8  -0.453649083002  -0.528343815383  stand
  9  -0.490939416039  -0.568575043788  stand
 10  -0.549829835508  -0.607098653851  stand
4202000000 hard 16
  7  -0.383823569133  -0.512770855539  stand
  8  -0.470348161953  -0.524548565445  stand
  9  -0.479820652027  -0.563340459880  stand
 10  -0.549523269319  -0.613102665910  stand
4300010000 hard 16
  7  -0.410142343948  -0.472378708454  stand
  8  -0.465602463334  -0.468207289024  stand
  9  -0.472949247765  -0.494738405856  stand
 10  -0.559354995543  -0.559438727278  stand
4401000000 hard 16
  7  -0.385450728134  -0.535070791425  stand
  8  -0.458130081301  -0.526427231305  stand
  9  -0.456794425087  -0.545180023229  stand
 10  -0.555981416957  -0.606303046547  stand
3200000010 hard 16
  7  -0.447322180561  -0.441213084304  hit
  8  -0.472223354470  -0.456703463340  hit
  9  -0.479937805519  -0.493098611647  stand
 10  -0.536245114998  -0.546487910240  stand
3210010000 hard 16
  7  -0.423622232372  -0.461907974092  stand
  8  -0.476115277236  -0.475282923968  hit
  9  -0.499310011222  -0.511472321874  stand
 10  -0.559782762318  -0.562247163768  stand
3300001000 hard 16
  7  -0.430808367280  -0.461713650313  stand
  8  -0.484885930859  -0.466284533202  hit
  9  -0.493772912661  -0.496445575600  stand
 10  -0.534350425788  -0.558828609997  stand
3400100000 hard 16
  7  -0.416830078600  -0.526579008059  stand
  8  -0.471334035599  -0.524614013326  stand
  9  -0.470042946277  -0.546313206551  stand
 10  -0.563243183452  -0.611536281179  stand
4000101000 hard 16
  7  -0.433412645211  -0.440129359937  stand
  8  -0.486248106214  -0.457834523830  hit
  9  -0.504211550723  -0.510427015475  stand
 10  -0.516299204671  -0.567786745722  stand
4003000000 hard 16
  7  -0.385135407228  -0.493356279757  stand
  8  -0.481202962286  -0.523364317664  stand
  9  -0.501794953014  -0.579797693326  stand
 10  -0.546155583875  -0.618570028950  stand
4011100000 hard 16
  7  -0.402023189516  -0.497493195026  stand
  8  -0.472235563642  -0.520460932053  stand
  9  -0.499812156080  -0.572476261685  stand
 10  -0.543789732672  -0.616606875824  stand
4040000000 hard 16
  7  -0.390400146271  -0.514364809445  stand
  8  -0.436029578287  -0.531759773796  stand
  9  -0.504734894525  -0.573667096753  stand
 10  -0.544429138644  -0.601127781275  stand
4100200000 hard 16
  7  -0.417986853381  -0.501545608863  stand
  8  -0.477829506474  -0.514249250466  stand
  9  -0.485342035541  -0.559328230854  stand
 10  -0.542924177354  -0.620083062617  stand
4110001000 hard 16
  7  -0.419337610773  -0.452973935688  stand
  8  -0.471060986659  -0.462724000805  hit
  9  -0.499513815736  -0.504868932826  stand
 10  -0.524353411761  -0.558363443609  stand
4200000100 hard 16
  7  -0.422783871451  -0.450885076353  stand
  8  -0.484912327227  -0.454764728559  hit
  9  -0.461603720537  -0.492138179461  stand
 10  -0.529648645928  -0.552049954205  stand
4210100000 hard 16
  7  -0.402330983443  -0.517335533629  stand
  8  -0.459021419118  -0.521742980974  stand
  9  -0.478730788969  -0.554947340427  stand
 10  -0.550765740216  -0.610784512330  stand
4320000000 hard 16
  7  -0.387157151791  -0.535165148644  stand
  8  -0.440453855088  -0.530757632041  stand
  9  -0.470789779326  -0.550230799011  stand
 10  -0.554645760743  -0.600102742786  stand
```

21.13 Stand with five or more cards vs. 7, six or more vs. 8, five or more vs. 9, and three or more vs. T; otherwise hit.

21.14 See Table B.26.

21.15 0.872820513; 0.126097208; 0.001026872; 0.000055407; mean: 1.128317173; H_0: -0.000365603.

21.17 $m = 2$.

21.18

21.19

21.20 (a) $-0.0\overline{6}$. (b) $0.27\overline{2}$.

Table B.26 Comparing two hard 16s vs. 9; distribution of dealer's final total (Problem 21.14).

draw	17	18	19	20	21	bust
A2229:						
0	.121267	.081408	.391473	.109818	.045902	.250132
1	.121622	.083502	.400408	.090328	.047229	.256912
2	.119776	.078654	.398001	.111038	.038821	.253709
3	.119998	.079319	.395885	.110541	.046160	.248096
4	.121240	.079631	.396390	.108293	.045473	.248973
5	.120661	.079852	.397800	.108814	.043241	.249632
A3336:						
0	.118672	.103431	.380995	.106115	.064079	.226708
1	.118915	.106085	.389602	.086544	.066313	.232540
2	.117629	.101195	.387401	.106683	.057861	.229231
3	.117470	.102248	.385214	.106755	.064642	.223672
4	.118531	.102135	.386419	.104332	.064545	.224039
5	.120590	.102185	.386817	.105554	.062155	.222699

21.21 If $p \geq 1/3$, take maximal insurance $f/2$; if $1/[3(1+f)] < p < 1/3$, take partial insurance $[3p(1+f)-1]/2$; if $p \leq 1/[3(1+f)]$, take no insurance.
21.22 Surrender with $\{7,9\}$ vs. T, $\{6,\text{T}\}$ vs. T, $\{6,9\}$ vs. T, $\{5,\text{T}\}$ vs. T, $\{7,7\}$ vs. T, and $\{6,\text{T}\}$ vs. A.
21.23
21.25 EoRs: A: 0.411084; 2: 0.286482; 3: 0.871192; 4: 1.829157; 5: 2.681044; 6: −1.709628; 7: 0.701684; 8: −0.038728; 9: −0.724774; T: −1.355503. No. Weights are $a, b, b, b, b, c, b, b, b, d$, where a, b, c, d are 0.083333333, 0.081481481, 0.061111111, 0.285185185.
21.26 EoRs: A: 0.787469; 2: 0.903884; 3: 1.250674; 4: 2.100667; 5: 2.482164; 6: −1.600394; 7: −1.955592; 8: 0.215158; 9: −0.387773; T: −0.949064. Index: 3.713147.
21.27

Chapter 22

22.2 $a/b > 3.4$.
22.3 $a/b > 31.2\overline{4}$.

22.4 five aces: 1; straight flush: 204; four of a kind: 828; full house: 4,368; flush: 7,804; straight: 20,532; three of a kind: 63,360; two pair: 138,600; one pair: 1,215,024; no pair: 1,418,964.

22.5 23,461,899.3 to 1.

22.6 2,225,270,496/463,563,500,400 or 0.004800357436.

22.7 (a) 1,024; 5,120; 15,360; 35,840; 71,680; 129,024; 215,040; 337,920; 506,880; 1,281,072. (b) 2,304; 7,680; 20,480; 44,800; 86,016; 150,528; 245,760; 380,160; 563,200; 1,368,757. (c) 4,080; 14,280; 34,680; 70,380; 127,500; 213,180; 335,580; 503,880; 1,295,400.

22.9 If $1 \le i \le n$ and $P \ge (2a + b_1 + \cdots + b_i)/(2a + 2b_1 + \cdots + 2b_i)$, then $(i, 0)$ is a saddle point, and this accounts for all saddle points.

22.10 If $P < P_n$, player 2 should fold at each round with probability $r_k = \frac{1}{2}$, and player 1 should bet with a losing hand at round 1 with probability $((3/2)^n - 1)P/(1 - P)$ and at round $k \ge 2$ with probability $((3/2)^{n-k+1} - 1)/((3/2)^{n-k+2} - 1)$. The game value is $v = 2(3/2)^n P$.

If $P \ge P_n$, then we can take $\boldsymbol{q}^* = (1, 0, \ldots, 0)$, $p_0^* = 0$, and

$$\sum_{i=k+1}^{n} p_i^* = \left(\left[\left(\frac{3}{2}\right)^{n-k} - 1\right]\frac{P}{1-P}\right) \wedge 1.$$

The game value is 2.

22.11 It is clear that player 1 should never check with a 3 and player 2 should never fold with a 3. This reduces the game to

$$\frac{1}{6} \times \begin{array}{c} \\ \text{CCB} \\ \text{CBB} \\ \text{BCB} \\ \text{BBB} \end{array} \begin{array}{c} \begin{array}{cccc} \text{FFC} & \text{FCC} & \text{CFC} & \text{CCC} \end{array} \\ \begin{pmatrix} 6a & 6a+b & 6a+b & 6a+2b \\ 6a-b & 6a & 6a+b & 6a+2b \\ 8a-b & 6a-b8a & 8a & 6a \\ 8a-2b & 4a-2b & 6a & 6a \end{pmatrix} \end{array}.$$

The reduces to a 2×2. CCB:FFC is a saddle point if $2a \le b$, and the game value is a. If $2a > b$, then the betting and calling probabilities are $p^* = b/(2a + b)$ and $q^* = (2a - b)/(2a + b)$, and the game value is $a + (b/6)(2a - b)/(2a + b)$.

22.12 (a) For each card player 1 holds, he has three strategies: 1 = check/fold, 2 = check/call, 3 = bet/—. For each card player 2 holds, he has four strategies, each depending on what player 1 did: 1 = check or fold, 2 = check or call, 3 = bet or fold, 4 = bet or call. A priori, we have a $3^3 \times 4^3$ matrix game. But players 1 and 2 should never fold with a 3 or call with a 1, and neither player should bet with a 2. This leads to

$$\frac{1}{6} \times \begin{array}{c} \\ (1,1,2) \\ (1,1,3) \\ (1,2,2) \\ (1,2,3) \\ (3,1,2) \\ (3,1,3) \\ (3,2,2) \\ (3,2,3) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1,4) & (1,2,4) & (3,1,4) & (3,2,4) \end{array} \\ \begin{pmatrix} 6a & 6a & 4a+b & 4a+b \\ 6a & 6a+b & 4a & 4a+b \\ 6a-b & 6a-b & 6a+b & 6a+b \\ 6a-b & 6a & 6a & 6a+b \\ 8a-b & 6a-2b & 6a & 4a-b \\ 8a-b & 6a-b & 6a-b & 4a-b \\ 8a-2b & 6a-3b & 8a & 6a-b \\ 8a-2b & 6a-2b & 8a-b & 6a-b \end{pmatrix} \end{array}$$

(b)

22.13 There are four saddle points, $(1,1)$ or $(1,3)$ for player 1 vs. $(1,2)$ or $(1,3)$ for player 2. The value of the game is 0.

22.14 With $S = 10$, the payoff matrix, multiplied by $52 \cdot 51$, is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	−1512	−2704	−3576	−4128	−4360	−4272	−3864	−3136	−2088	−720	968	2976	5304
2	1716	180	−1204	−2268	−3012	−3436	−3540	−3324	−2788	−1932	−756	740	2556	4692
3	3112	1704	296	−960	−1896	−2512	−2808	−2784	−2440	−1776	−792	512	2136	4080
4	4188	2908	1628	348	−780	−1588	−2076	−2244	−2092	−1620	−828	284	1716	3468
5	4944	3792	2640	1488	336	−664	−1344	−1704	−1744	−1464	−864	56	1296	2856
6	5380	4356	3332	2308	1284	260	−612	−1164	−1396	−1308	−900	−172	876	2244
7	5496	4600	3704	2808	1912	1016	120	−624	−1048	−1152	−936	−400	456	1632
8	5292	4524	3756	2988	2220	1452	684	−84	−700	−996	−972	−628	36	1020
9	4768	4128	3488	2848	2208	1568	928	288	−352	−840	−1008	−856	−384	408
10	3924	3412	2900	2388	1876	1364	852	340	−172	−684	−1044	−1084	−804	−204
11	2760	2376	1992	1608	1224	840	456	72	−312	−696	−1080	−1312	−1224	−816
12	1276	1020	764	508	252	−4	−260	−516	−772	−1028	−1284	−1540	−1644	−1428
13	−528	−656	−784	−912	−1040	−1168	−1296	−1424	−1552	−1680	−1808	−1936	−2064	−2040
14	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652	−2652

Eliminate columns 1–5 and 14, rows 11–14, column 13. Optimal strategy for player 2 is $q_{10}^* = 3/16$ and $q_{11}^* = 13/16$. Optimal strategies for player 1 include the following 16 strategies. (See `http://banach.lse.ac.uk/form.html`.)

1. $p_1^* = 7/64$, $p_9^* = 57/64$.
2. $p_2^* = 1/8$, $p_9^* = 7/8$.
3. $p_3^* = 7/48$, $p_9^* = 41/48$.
4. $p_4^* = 7/40$, $p_9^* = 33/40$.
5. $p_5^* = 7/32$, $p_9^* = 25/32$.
6. $p_6^* = 7/24$, $p_9^* = 17/24$.
7. $p_7^* = 7/16$, $p_9^* = 9/16$.
8. $p_8^* = 7/8$, $p_9^* = 1/8$.
9. $p_1^* = 5/24$, $p_{10}^* = 19/24$.
10. $p_2^* = 15/64$, $p_{10}^* = 49/64$.
11. $p_3^* = 15/56$, $p_{10}^* = 41/56$.
12. $p_4^* = 5/16$, $p_{10}^* = 11/16$.
13. $p_5^* = 3/8$, $p_{10}^* = 5/8$.
14. $p_6^* = 15/32$, $p_{10}^* = 17/32$.
15. $p_7^* = 5/8$, $p_{10}^* = 3/8$.
16. $p_8^* = 15/16$, $p_{10}^* = 1/16$.

The value of the game (to player 1) is approximately -0.368213. For general S, see the masters project at the University of Utah by Julie Billings (August 2010).

22.15 (a) 0.0130612. (b) 0.0906122. (c) 0.187755. (d) 0.00653061.

22.16 (a) 0.0128571. (b) 0.00821429. (c) 0.000204082. (d) 0.0212755. (e) 0.0893367. (f) 0.180000. (g) 0.00642857. (h) 0.109439. (i) 0.00704082. (j) 0.0187755.

22.17 (a) By the turn. Unpaired hole cards: 0.589453; 0.345046; 0.0369692; 0.0246461; 0.00347807; 0.000408163. Pocket pairs: 0.844898; 0.150204; 0.00489796. Suited hole cards: 0.357148; 0.436513; 0.176965; 0.0293747. (b) By the river. Unpaired hole cards: 0.512568; 0.384426; 0.0562574; 0.0375050; 0.00822368; 0.00102041. Pocket pairs: 0.808163; 0.183673; 0.00816327. Suited hole cards: 0.271742; 0.427024; 0.237235; 0.639983.

22.18 K♡-K♠wins with probability $0.5\overline{76}$, loses with probability $0.4\overline{17}$, and ties with probability $0.0\overline{06}$.

22.19 (a) 0.371068. (b) 0.372978. (c) .370481.

22.20 Flop, 2-2 to K-K: 1.000000; 0.998980; 0.993878; 0.981429; 0.958367; 0.921429; 0.867347; 0.792857; 0.694694; 0.569592; 0.414286; 0.225510. Board, 2-2 to K-K: 1.000000; 0.999997; 0.999881; 0.999055; 0.995956; 0.987571; 0.968954; 0.932741; 0.868670; 0.763096; 0.598507; 0.353040.

22.21 1: 0.155102, 0.117551, 0.079184; 2: 0.291424, 0.225510, 0.155102; 3: 0.410547, 0.324286, 0.227755; 4: 0.513982, 0.414286, 0.297143; 5: 0.603170, 0.495918, 0.363265; 6: 0.679483, 0.569592, 0.426122; 7: 0.744225, 0.635714, 0.485714; 8: 0.798628, 0.694694, 0.542041; 9: 0.843856, 0.746939, 0.595102.

22.22 For ten hands, 0.132829; 0.366423; 0.018321; 0.329781; 0.021276; 0.113473; 0.000166; 0.005319; 0.012411. For nine hands, 0.171303; 0.397865; 0.018650; 0.298399; 0.018085; 0.084396; 0.000133; 0.003723; 0.007447.

22.23 0.002158999415.

22.24 See Table B.27.

22.25 0.264132; 0.265855; 0.470014. $r = 0$: 37; 10; 9. $r = 1$: 1,640; 920; 240. $r = 2$: 26,690; 14,954; 2,036. $r = 3$: 152,297; 84,486; 39,857. $r = 4$: 252,953; 296,107; 182,060. $r = 5$: 18,657; 58,747; 580,604.

22.26 (a) A-9 with one suit match; 0.052093; 0.933704; 0.014203. (b) 6-5 suited with no suit match; 0.228687; 0.767566; 0.003746.

22.27 (a) K-K vs. K-2 unsuited maximizes expectation: 0.892532. (b) K♠-K♡ vs. K♢-2♠ maximizes expectation: 0.898475. (c) 6-6 vs. 9-8 suited minimizes expectation: 0.000123226. (d) 3♠-3♡ vs. A♠-10♠ minimizes expectation: 0.000141330.

22.28 247.

22.29 $E_{\text{A-Ku,J-Ts}} \approx 0.189722$, $E_{\text{J-Ts,2-2}} \approx 0.076768$, and $E_{\text{2-2,A-Ku}} \approx 0.052983$.

22.30 7,108.

22.31 ∞; 478.008197; 554.509992; 331.887184.

Table B.27 Probability of a better ace. Here m (row number) is the number of better denominations and n (column number) is the number of opponents (Problem 22.24).

	n								
m	9	8	7	6	5	4	3	2	1
0	.022041	.019592	.017143	.014694	.012245	.009796	.007347	.004898	.002449
1	.107711	.096020	.084259	.072429	.060531	.048563	.036526	.024420	.012245
2	.188481	.168624	.148499	.128105	.107439	.086502	.065290	.043804	.022041
3	.264486	.237496	.209920	.181752	.152986	.123617	.093640	.063048	.031837
4	.335861	.302726	.268578	.233404	.197188	.159917	.121577	.082154	.041633
5	.402740	.364403	.324530	.283092	.240061	.195408	.149104	.101120	.051429
6	.465261	.422619	.377832	.330850	.281621	.230095	.176221	.119948	.061224
7	.523557	.477462	.428540	.376708	.321884	.263986	.202931	.138637	.071020
8	.577765	.529024	.476710	.420699	.360866	.297086	.229235	.157186	.080816
9	.628019	.577394	.522400	.462856	.398584	.329403	.255134	.175597	.090612
10	.674456	.622664	.565665	.503211	.435053	.360943	.280631	.193869	.100408
11	.717209	.664922	.606562	.541796	.470290	.391711	.305726	.212002	.110204