1 Introduction to modeling

Roughly defined, mathematical modeling is the process of constructing mathematical objects whose behaviors or properties correspond in some way to a particular real-world system. In this description, a mathematical object could be a system of equations, a stochastic process, a geometric or algebraic structure, an algorithm, or even just a set of numbers. The term real-world system could refer to a physical system, a financial system, a social system, an ecological system, or essentially any other system whose behaviors can be observed.

1.1 Why model?

What is the motivation behind mathematical modeling? There are of course many specific reasons, but most are related in some way to the following two:

- **To gain understanding.** Generally speaking, if we have a mathematical model which accurately reflects some behavior of a real-world system of interest, we can often gain improved understanding of that system through analysis of the model. Furthermore, in the process of building the model we find out which factors are most important in the system, and how different parts of the system are related.

- **To predict or simulate.** Very often we wish to know what a real-world system will do in the future, but it is expensive, impractical, or impossible to experiment directly with the system. Examples include nuclear reactor design, space flight, extinction of species, weather prediction, drug efficacy in humans, and so on.

It should be apparent that much of modern science involves mathematical modeling. The old adage “mathematics is the language of science” is really true. Scientists use mathematics to describe real phenomena, and in fact much of this activity constitutes mathematical modeling. As computers become cheaper and powerful and their use becomes more widespread, mathematical models play an increasingly important role in science.

From a business perspective, it is clear that an improved ability to simulate, predict, or understand certain real-world systems through mathematical modeling provides a distinct competitive advantage. Examples: the stock market, aircraft design, oil production, semiconductor manufacturing. Furthermore, just as in pure science, as computing power becomes
cheaper, modeling becomes an increasingly cost-effective alternative to direct experimentation.

1.2 The modeling process

So modeling is important. How does one do it? Unfortunately, there is no definite “algorithm” to construct a mathematical model that will work in all situations. Modeling is sometimes viewed as an art. It involves taking whatever knowledge you may have of mathematics and of the system of interest and using that knowledge to create something. Since everyone has a different knowledge base, a preferred bag of tricks, and a unique way of looking at problems, different people may come up with different models for the same system. There is usually plenty of room for argument about which model is “best”.

It is very important to understand at the outset that for any real system, there is no “perfect” model\(^1\). One is always faced with tradeoffs between

- accuracy,
- flexibility,
- cost.

Increasing the accuracy of a model generally increases cost and decreases flexibility. The goal in creating a model is usually to obtain a “sufficiently accurate” and flexible model at a low cost.

Note that this situation is much different than what we normally encounter when we solve purely mathematical problems, at least in the context of the traditional mathematical curriculum. Usually in a mathematics text we find very precise and explicit problems, which we are asked to solve completely. We may have to struggle to find a solution, but when we finally finish a problem, it is done. In modeling on the other hand, we are faced with unclearly stated and ambiguous problems which we can never hope to solve completely! It may sound awful but actually it can be a lot of fun. And it is much more representative of how things are done in the “real world” (even in mathematical research).

\(^1\)This statement can be challenged, but “perfect” is meant here in a strict sense. One may object with possible counterexamples from physics, but since there is not yet a “Theory of Everything”, it can be asserted that no existing mathematical model provides a perfectly accurate and complete picture of reality.
One of the most useful ways to view modeling is as a process, as illustrated in Figure 1.1. Notice that the figure represents a loop, or an iterative process. The starting point is the bubble in the upper left-hand corner, real world data. This could represent quantitative measurements of the system of interest, general knowledge about how it works, or both. In any case we need some information pertaining to the system; usually the more information, the better. From that information we proceed to formulate, or construct, a model. Constructing a model requires:

- A clear picture of the goal of the modeling exercise. Exactly which aspects of the system do you wish to understand or predict, and how accurately do you need to do it?

- An picture of the key factors involved in the system and how they relate to each other. This often requires taking a greatly simplified view of the system, neglecting factors known to influence the system, and making assumptions which may or may not be correct.

A model may consist of algebraic, differential, or integral equations, stochastic processes, geometrical structures, etc.

The following situations are very common in modeling:

- Good models already exist for parts of the system. The goal is then to assemble these “submodels” to represent the whole system of interest.
• Good models already exist for a different system, which can be translated or modified to apply to the system of interest. One of the greatest virtues of mathematics is its generality.

• A general model exists which includes the system of interest as a special case, but it is very difficult to compute with or analyze the general model. The goal is then to simplify or make approximations to the general model which will still reflect the behavior of the particular system of interest. Situations like this occur often in fluid dynamics, where the Navier–Stokes equations are capable of modeling an extremely wide range of fluid flow problems, but it is usually not feasible to solve the equations in full generality.

The three situations above illustrate the importance of proper background research in modeling. It is often astounding how much information and previous modeling work one can uncover by searching the scientific literature, the internet, government reports, and so on. In very lucky cases you may even find software which can be used for your problem.

Once a model is constructed, one generally needs to do some analysis or computation to make it produce results. The results are often approximate solutions to the equations of interest, but for example in the case of stochastic models, results could be in the form of statistical information.

Since the model is supposed to somehow represent the real system, it should be possible to interpret results from the model in terms of observable properties of the system. This is not always a trivial step, but it must be done if the results are to be of any use. Interpretation of the results should lead to predictions or explanations about the real system, which can then be tested against real observations to determine the effectiveness of the model. If the observations do not agree sufficiently well with predictions from the model, the loop is iterated again with a revised formulation, new results, predictions, and so on.

The description above of the modeling process should remind you of scientific method, which has been the foundation of scientific research for hundreds of years. The scientific method goes something like this:

1. Make general observations of phenomena,
2. Formulate a hypothesis,
3. Develop a method to test hypothesis,
4. Obtain data,
5. Test hypothesis against data,

6. Attempt to confirm or deny hypothesis.

Mathematical models are often used in this context, where the hypothesis is a mathematical model. However as we discussed above, mathematical modeling can also be used in situations outside of scientific research. In nonscientific applications, the goal may be simply to obtain something which can produce answers “close enough” for a particular purpose, with the clear understanding that important features of the system may have been neglected.

In this course, our goal will be to learn by way of direct experience, something about the process of mathematical modeling.