The Strong Markov Property

Throughout, $X := \{X_t\}_{t\geq 0}$ denotes a Lévy process on $\mathbf{R}^d$ with triple $(a\,,\sigma, m)$, and exponent Ψ. And from now on, we let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration of X, all the time remembering that, in accord with our earlier convention, $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions.

Transition measures and the Markov property

Definition 1. The *transition measures* of X are the probability measures

$$P_t(x\,,A) := \mathrm{P}\,\{x + X_t \in A\}$$

defined for all $t \geq 0$, $x \in \mathbf{R}^d$, and $A \in \mathcal{B}(\mathbf{R}^d)$. In other words, each $P_t(x\,,\bullet)$ is the law of X_t started at $x \in \mathbf{R}^d$. We single out the case $x = 0$ by setting $\mu_t(A) := P_t(0\,,A)$; thus, μ_t is the distribution of X_t for all $t > 0$. □

Note, in particular, that $\mu_0 = \delta_0$ is the point mass at $0 \in \mathbf{R}^d$.

Proposition 2. *For all $s, t \geq 0$, and measurable $f : \mathbf{R}^d \to \mathbf{R}_+$,*

$$\mathrm{E}[f(X_{t+s}) \,|\, \mathcal{F}_s] = \int_{\mathbf{R}^d} f(y)\, P_t(X_s\,, \mathrm{d}y) \qquad \text{a.s.}$$

Consequently, for all $x_0 \in \mathbf{R}^d$, $0 < t_1 < t_2 < \cdots < t_k$, and measurable $f_1, \ldots, f_k : \mathbf{R}^d \to \mathbf{R}_+$,

$$\begin{aligned} &\mathrm{E}\left(\prod_{j=1}^k f_j(x_0 + X_{t_j})\right) \qquad (1)\\ &= \int_{\mathbf{R}^k} P_{t_1}(x_0\,, \mathrm{d}x_1) \int_{\mathbf{R}^k} P_{t_2 - t_1}(x_1\,, \mathrm{d}x_2) \cdots \int_{\mathbf{R}^d} P_{t_k - t_{k-1}}(x_{k-1}\,, \mathrm{d}x_k)\ \prod_{j=1}^k f_j(x_j). \end{aligned}$$

Property (1) is called the *Chapman–Kolmogorov equation*. That property has the following ready consequence: Transition measures determine the finite-dimensional distributions of X uniquely.

Definition 3. Any stochastic process $\{X_t\}_{t\geq 0}$ that satisfies the Chapman-Kolmogorov equation is called a *Markov process*. This definition continues to make sense if we replace $(\mathbf{R}^d\,, \mathscr{B}(\mathbf{R}^d))$ by any measurable space on which we can construct infinite families of random variables. □

Thus, Lévy processes are cadlag Markov processes that have special "addition" properties. In particular, as Exercise below 1 shows, Lévy processes have the important property that the finite-dimensional distributions of X are described not only by $\{P_t(x\,,\cdot)\}_{t\geq 0, x\in\mathbf{R}^d}$ but by the much-smaller family $\{\mu_t(\cdot)\}_{t\geq 0}$.

Note, in particular, that if $f : \mathbf{R}^d \to \mathbf{R}_+$ is measurable, $t \geq 0$, and $x \in \mathbf{R}^d$, then

$$\mathrm{E}f(x + X_t) = \int_{\mathbf{R}^d} f(y)\, P_t(x\,, \mathrm{d}y) = \int_{\mathbf{R}^d} f(x + y)\, \mu_t(\mathrm{d}y).$$

Therefore, if we define

$$\tilde{\mu}_t(A) := \mu_t(-A) \qquad \text{for all } t \geq 0 \text{ and } A \in \mathscr{B}(\mathbf{R}^d),$$

where $-A := \{-a : a \in A\}$, then we have the following convolution formula, valid for all measurable $f : \mathbf{R}^d \to \mathbf{R}_+$, $x \in \mathbf{R}^d$, and $t \geq 0$:

$$\mathrm{E}f(x + X_t) = (f * \tilde{\mu}_t)(x).$$

And, more generally, for all measurable $f : \mathbf{R}^d \to \mathbf{R}_+$, $x \in \mathbf{R}^d$, and $s, t \geq 0$

$$\mathrm{E}[f(X_{t+s}) \mid \mathscr{F}_s] = (f * \tilde{\mu}_t)(X_s) \qquad \text{a.s.}$$

[Why is this more general?]

Proposition 4. *The family $\{\mu_t\}_{t\geq 0}$ of Borel probability measure on $\mathbf{R}^d$ is a "convolution semigroup" in the sense that $\mu_t * \mu_s = \mu_{t+s}$ for all $s, t \geq 0$. Moreover, $\hat{\mu}_t(\xi) = \exp(-t\Psi(\xi))$ for all $t \geq 0$ and $\xi \in \mathbf{R}^d$. Similarly, $\{\tilde{\mu}_t\}_{t\geq 0}$ is a convolution semigroup with $\hat{\tilde{\mu}}_t(\xi) = \exp(-t\Psi(-\xi))$.*

Proof. The assertion about $\tilde{\mu}$ follows from the assertion about μ [or you can repeat the following with $\tilde{\mu}$ in place of μ].

Since μ_t is the distribution of X_t, the characteristic function of X_t is described by $\hat{\mu}_t(\xi) = \exp(-t\Psi(\xi))$. The proposition follows immediately from this, because $\hat{\mu}_t(\xi) \cdot \hat{\mu}_s(\xi) = \exp(-(t+s)\Psi(\xi)) = \hat{\mu}_{t+s}(\xi)$. □

The strong Markov property

Theorem 5 (The strong Markov property)**.** *Let T be a finite stopping time. Then, the process $X^T := \{X_t^T\}_{t\geq 0}$, defined by $X_t^T := X_{T+t} - X_T$ is a Lévy process with exponent Ψ and independent of $\mathcal{F}_T$.*

Proof. X^T is manifestly cadlag [because X is]. In addition, one checks that whenever $0 < t_1 < \cdots < t_k$ and $A_1, \ldots, A_k \in \mathcal{B}(\mathbf{R}^d)$,

$$\mathrm{P}\left(\bigcap_{j=1}^{k}\left\{X_{T+t_j} - X_T \in A_j\right\} \,\middle|\, \mathcal{F}_T\right) = \mathrm{P}\left(\bigcap_{j=1}^{k}\left\{X_{t_j} \in A_j\right\}\right) \qquad \text{a.s.;}$$

see Exercise 2 on page 32. This readily implies that the finite-dimensional distributions of X^T are the same as the finite-dimensional distributions of X, and the result follows. □

Theorem 5 has a number of deep consequences. The following shows that Lévy processes have the following variation of strong Markov property. The following is attractive, in part because it can be used to study processes that do not have good additivity properties.

Corollary 6. *For all finite stopping times T, every $t \geq 0$, and all measurable functions $f : \mathbf{R}^d \to \mathbf{R}_+$.*

$$\mathrm{E}\left[f(X_{T+t}) \mid \mathcal{F}_T\right] = \int_{\mathbf{R}^d} f(y) P_t(X_T\,, \mathrm{d}y) \qquad \textit{a.s.}$$

Let T be a finite stopping time, and then define $\mathcal{F}^{(T)} = \{\mathcal{F}_t^{(T)}\}_{t\geq 0}$ to be the natural filtration of the Lévy process X^T. The following is a useful corollary of the strong Markov property.

Corollary 7 (Blumenthal's zero-one law; Blumenthal, 1957)**.** *Let T be a finite stopping time. Then $\mathcal{F}_0^{(T)}$ is trivial; i.e., $\mathrm{P}(A) \in \{0\,, 1\}$ for all $A \in \mathcal{F}_0^{(T)}$.*

The following are nontrivial examples of elements of $\mathcal{F}_0^{(T)}$:

$$\Upsilon_1 := \left\{ \liminf_{t\downarrow 0} \frac{\|X_{T+t} - X_T\|}{t^{1/\alpha}} = 0 \right\} \qquad \text{where } \alpha > 0 \text{ is fixed;}$$

$$\Upsilon_2 := \left\{ \limsup_{t\downarrow 0} \frac{\|X_{T+t} - X_T\|}{\sqrt{2t\ln\ln(1/t)}} = 1 \right\}; \text{ or}$$

$$\Upsilon_3 := \left\{ {}^{\exists} t_n \downarrow 0 \text{ such that } X_{T+t_n} - X_T > 0 \text{ for all } n \geq 1 \right\} \text{ in dimension one, etc.}$$

Proof of Blumenthal's zero-one law. The strong Markov property [Corollary 6] reduces the problem to $T \equiv 0$. And of course we do not need to write $\mathcal{F}^{(0)}$ since $\mathcal{F}_t^{(0)}$ is the same object as $\mathcal{F}_t$.

For all $n \geq 1$ define $\mathcal{A}_n$ to be the completion of the sigma-algebra generated by the collection $\{X_{t+2^{-n}} - X_{2^{-n}}\}_{t\in[0,2^{-n}]}$. By the Markov property, $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are independent sigma-algebras. Their tail sigma-algebra $\mathcal{T}$ is the smallest sigma-algebra that contains $\cup_{i=N}^{\infty}\mathcal{A}_i$ for all $N \geq 1$. Clearly $\mathcal{T}$ is complete, and Kolmogorov's zero-one law tells us that $\mathcal{T}$ is trivial. Because $\cup_{i=N}^{\infty}\mathcal{A}_i$ contains the sigma-algebra generated by all increments of the form $X_{u+v} - X_u$ where $u, v \in [2^{-m}, 2^{-m+1}]$ for some $m \geq N$, and since $X_u \to 0$ as $u \downarrow 0$, it follows that $\mathcal{T}$ contains $\cap_{s\geq 0}\mathcal{X}_s$, where $\mathcal{X}_s$ denotes the sigma-algebra generated by $\{X_r\}_{r\in[0,s]}$. Since $\mathcal{T}$ is complete, this implies $\mathcal{F}_0 \subseteq \mathcal{T}$ [in fact, $\mathcal{T} = \mathcal{F}_0$] as well, and hence $\mathcal{F}_0$ is trivial because $\mathcal{T}$ is. □

Thus, for example, take the set Υ_1 introduced earlier. We can apply the Blumenthal zero-one to the $\Upsilon_1 \in \mathcal{D}_0$ and deduce the following:

$$\text{For every } \alpha > 0, \qquad \mathrm{P}\left\{ \liminf_{t\downarrow 0} \frac{\|X_{T+t} - X_T\|}{t^{1/\alpha}} = 0 \right\} = 0 \text{ or } 1.$$

You should construct a few more examples of this type.

Feller semigroups and resolvents

Define a collection $\{P_t\}_{t\geq 0}$ of linear operators by

$$(P_t f)(x) := \mathrm{E}f(x + X_t) = \int_{\mathbf{R}^d} f(y)P_t(x\,, \mathrm{d}y) = (f * \tilde{\mu}_t)(x) \quad \text{for } t \geq 0,\ x \in \mathbf{R}^d.$$

[Since $X_0 = 0$, $P_0 = \delta_0$ is point mass at zero.] The preceding is well defined for various measurable functions $f : \mathbf{R}^d \to \mathbf{R}$. For instance, everything is fine if f is nonnegative, and also if $(P_t|f|)(x) < \infty$ for all $t \geq 0$ and $x \in \mathbf{R}^d$ [in that case, we can write $P_t f = P_t f^+ - P_t f^-$].

The Markov property of X [see, in particular, Proposition 4] tells us that $(P_{t+s}f)(x) = (P_t(P_s f))(x)$. In other words,

$$P_{t+s} = P_t P_s = P_s P_t \qquad \text{for all } s, t \geq 0, \tag{2}$$

where P_tP_sf is shorthand for $P_t(P_sf)$ etc. Since P_t and P_s commute, in the preceding sense, there is no ambiguity in dropping the parentheses.

Definition 8. The family $\{P_t\}_{t\geq 0}$ is the *semigroup* associated with the Lévy process X. The *resolvent* $\{R_\lambda\}_{\lambda>0}$ of the process X is the family of linear operators defined by

$$(R_\lambda f)(x) := \int_0^\infty \mathrm{e}^{-\lambda t}(P_tf)(x)\,\mathrm{d}t = \mathrm{E}\int_0^\infty \mathrm{e}^{-\lambda t} f(x+X_t)\,\mathrm{d}t \quad (\lambda > 0).$$

This can make sense also for $\lambda = 0$, and we write R in place of R_0. Finally, R_λ is called the *λ-potential of f* when $\lambda > 0$; when $\lambda = 0$, we call it the potential of f instead. □

Remark 9. It might be good to note that we can cast the strong Markov property in terms of the semigroup $\{P_t\}_{t\geq 0}$ as follows: For all $s \geq 0$, finite stopping times T, and $f : \mathbf{R}^d \to \mathbf{R}_+$ measurable, $\mathrm{E}[f(X_{T+s}) \mid \mathcal{F}_T] = (P_sf)(X_T)$ almost surely. □

Formally speaking,

$$R_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} P_t\,\mathrm{d}t \qquad (\lambda \geq 0)$$

defines the Laplace transform of the [infinite-dimensional] function $t \mapsto P_t$. Once again, $R_\lambda f$ is defined for all Borel measurable $f : \mathbf{R}^d \to \mathbf{R}$, if either $f \geq 0$; or if $R_\lambda|f|$ is well defined.

Recall that $C_0(\mathbf{R}^d)$ denotes the collection of all continuous $f : \mathbf{R}^d \to \mathbf{R}$ that vanish at infinity [$f(x) \to 0$ as $\|x\| \to \infty$]; $C_0(\mathbf{R}^d)$ is a Banach space in norm $\|f\| := \sup_{x\in\mathbf{R}^d} |f(x)|$.

The following are easy to verify:

(1) Each P_t is a contraction [more precisely nonexpansive] on $C_0(\mathbf{R}^d)$. That is, $\|P_tf\| \leq \|f\|$ for all $t \geq 0$;

(2) $\{P_t\}_{t\geq 0}$ is a *Feller semigroup*. That is, each P_t maps $C_0(\mathbf{R}^d)$ to itself and $\lim_{t\downarrow 0} \|P_tf - f\| = 0$;

(3) If $\lambda > 0$, then λR_λ is a contraction [nonexpansive] on $C_0(\mathbf{R}^d)$;

(4) If $\lambda > 0$, then λR_λ maps $C_0(\mathbf{R}^d)$ to itself.

The preceding describe the smoothness behavior of P_t and R_λ for fixed t and λ. It is also not hard to describe the smoothness properties of them as functions of t and λ. For instance,

Proposition 10. *For all $f \in C_0(\mathbf{R}^d)$,*

$$\lim_{t\downarrow 0}\sup_{s\geq 0} \|P_{t+s}f - P_sf\| = 0 \quad \textit{and} \quad \lim_{\lambda\uparrow\infty} \|\lambda R_\lambda f - f\| = 0.$$

Proof. We observe that

$$\|P_tf - f\| = \sup_{x\in\mathbf{R}^d} |\mathrm{E}f(x + X_t) - f(x)| \le \sup_{x\in\mathbf{R}^d} \mathrm{E}\,|f(x + X_t) - f(x)|\,.$$

Now every $f \in C_0(\mathbf{R}^d)$ is uniformly continuous and bounded on all of $\mathbf{R}^d$. Since X is right continuous and f is bounded, it follows from the bounded convergence theorem that $\lim_{t\downarrow 0}\|P_tf - f\| = 0$. But the semigroup property implies that $\|P_{t+s}f - P_sf\| = \|P_s(P_tf - f)\| \le \|P_tf - f\|$, since P_s is a contraction on $C_0(\mathbf{R}^d)$. This proves the first assertion. The second follows from the first, since $\lambda R_\lambda = \int_0^\infty \mathrm{e}^{-t} P_{t/\lambda}\,\mathrm{d}t$ by a change of variables. □

Proposition 11. *If $f \in C_0(\mathbf{R}^d)\cap L^p(\mathbf{R}^d)$ for some $p \in [1\,,\infty)$, then $\|P_tf\|_{L^p(\mathbf{R}^d)} \le \|f\|_{L^p(\mathbf{R}^d)}$ for all $t \ge 0$ and $\|\lambda R_\lambda f\|_{L^p(\mathbf{R}^d)} \le \|f\|_{L^p(\mathbf{R}^d)}$ for all $\lambda > 0$.*

In words, the preceding states that P_t and λR_λ are contractions on $L^p(\mathbf{R}^d)$ for every $p \in [1\,,\infty)$ and $t, \lambda > 0$.

Proof. If $f \in C_0(\mathbf{R}^d) \cap L^p(\mathbf{R}^d)$, then for all $t \ge 0$,

$$\begin{aligned}\int_{\mathbf{R}^d} |(P_tf)(x)|^p\,\mathrm{d}x = \int_{\mathbf{R}^d} |\mathrm{E}f(x + X_t)|^p\,\mathrm{d}x &\le \int_{\mathbf{R}^d} \mathrm{E}\left(|f(x + X_t)|^p\right)\mathrm{d}x \\ &= \int_{\mathbf{R}^d} |f(y)|^p\,\mathrm{d}y.\end{aligned}$$

This proves the assertion about P_t; the one about R_λ is proved similarly. □

The Hille–Yosida theorem

One checks directly that for all $\mu, \lambda \ge 0$,

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu. \tag{3}$$

This is called the *resolvent equation*, and has many consequences. For instance, the resolvent equation implies readily the commutation property $R_\mu R_\lambda = R_\lambda R_\mu$. For another consequence of the resolvent eqution, suppose $g = R_\mu f$ for some $f \in C_0(\mathbf{R}^d)$ and $\mu > 0$. Then, $g \in C_0(\mathbf{R}^d)$ and by the resolvent equation, $R_\lambda f - g = (\mu - \lambda)R_\lambda g$. Consequently, $g = R_\lambda h$, where $h := f + (\lambda - \mu)R_\lambda g \in C_0(\mathbf{R}^d)$. In other words, $R_\mu(C_0(\mathbf{R}^d)) = R_\lambda(C_0(\mathbf{R}^d))$, whence

$$\mathrm{Dom}[L] := \left\{R_\mu f : f \in C_0(\mathbf{R}^d)\right\} \quad \text{does not depend on } \mu > 0.$$

And Dom$[L]$ is dense in $C_0(\mathbf{R}^d)$ [Proposition 10].

For yet another application of the resolvent equation, let us suppose that $R_\lambda f \equiv 0$ for some $\lambda > 0$ and $f \in C_0(\mathbf{R}^d)$. Then the resolvent equation implies that $R_\mu f \equiv 0$ for all μ. Therefore, $f = \lim_{\mu\uparrow 0} \mu R_\mu f = 0$. This implies that every R_λ is a one-to-one and onto map from $C_0(\mathbf{R}^d)$ to Dom$[L]$; i.e., it is invertible!

Definition 12. The [infinitesimal] *generator* of X is the linear operator $L : \text{Dom}[L] \to C_0(\mathbf{R}^d)$ that is defined uniquely by

$$L := \lambda I - R_\lambda^{-1},$$

where $If := f$ defines the identity operator I on $C_0(\mathbf{R}^d)$. The space Dom$[L]$ is the *domain of* L. □

The following is perhaps a better way to think about L; roughly speaking, it asserts that $P_t f - f \simeq tLf$ for t small, and $\lambda R_\lambda f - f \simeq \lambda^{-1} Lf$ for λ large.

Theorem 13 (Hille XXX, Yosida XXX)**.** *If $f \in$ Dom$[L]$, then*

$$\lim_{\lambda\uparrow\infty} \sup_{x\in\mathbf{R}^d} \left| \frac{\lambda(R_\lambda f)(x) - f(x)}{1/\lambda} - (Lf)(x) \right| = \lim_{t\downarrow 0} \sup_{x\in\mathbf{R}^d} \left| \frac{(P_t f)(x) - f(x)}{t} - (Lf)(x) \right| = 0.$$

Because $f = P_0 f$, the Hille–Yosida theorem implies, among other things, that $(\partial/\partial t)P_t|_{t=0} = L$, where the partial derivative is really a right derivative. See Exercise 4 for a consequence in partial integro-differential equations.

Proof. Thanks to Proposition 10 and the definition of the generator, $Lf = \lambda f - R_\lambda^{-1} f$ for all $f \in$ Dom$[L]$, whence

$$\lambda R_\lambda Lf = \frac{\lambda R_\lambda f - f}{1/\lambda} \to Lf \qquad \text{in } C_0(\mathbf{R}^d) \text{ as } \lambda \uparrow \infty.$$

This proves half of the theorem. For the other half recall that Dom$[L]$ is the collection of all functions of the form $f = R_\lambda h$, where $h \in C_0(\mathbf{R}^d)$ and $\lambda > 0$. By the semigroup property, for such λ and h we have

$$\begin{aligned} P_t R_\lambda h &= \int_0^\infty \mathrm{e}^{-\lambda s} P_{t+s} h \,\mathrm{d}s = \mathrm{e}^{\lambda t} \int_t^\infty \mathrm{e}^{-\lambda s} P_s h \,\mathrm{d}s \\ &= \mathrm{e}^{\lambda t} \left(R_\lambda h - \int_0^t \mathrm{e}^{-\lambda s} P_s h \,\mathrm{d}s \right). \end{aligned}$$

Consequently, for all $f = R_\lambda h \in$ Dom$[L]$,

$$\begin{aligned} \frac{P_t f - f}{t} &= \left(\frac{\mathrm{e}^{\lambda t} - 1}{t} \right) R_\lambda h - \frac{\mathrm{e}^{\lambda t}}{t} \int_0^t \mathrm{e}^{-\lambda s} P_s h \,\mathrm{d}s \\ &\to \lambda R_\lambda h - h \qquad \text{in } C_0(\mathbf{R}^d) \text{ as } t \downarrow 0. \end{aligned}$$

But $\lambda R_\lambda h - h = \lambda f - R_\lambda^{-1} f = Lf$. □

The form of the generator

Let $\mathcal{S}$ denote the collection of all rapidly-decreasing test functions $f : \mathbf{R}^d \to \mathbf{R}$. That is, $f \in \mathcal{S}$ if and only if $f \in C^\infty(\mathbf{R}^d)$, and f and all of its partial derivatives vanish faster than any polynomial. In other words, if D is a differential operator [of finite order] and $n \geq 1$, then $\sup_{x\in\mathbf{R}^d}(1 + \|x\|^n)|(Df)(x)| < \infty$. It is easy to see that $\mathcal{S} \subset L^1(\mathbf{R}^d) \cap C_0(\mathbf{R}^d)$ and $\mathcal{S}$ is dense in $C_0(\mathbf{R}^d)$. And it is well known that if $f \in \mathcal{S}$, then $\hat{f} \in \mathcal{S}$ as well, and vice versa.

It is possible to see that if $f, \hat{f} \in L^1(\mathbf{R}^d)$, then for all $t \geq 0$ and $\lambda > 0$,

$$\widehat{P_t f}(\xi) = \mathrm{e}^{-t\Psi(-\xi)}\hat{f}(\xi), \quad \widehat{R_\lambda f}(\xi) = \frac{\hat{f}(\xi)}{\lambda + \Psi(-\xi)} \qquad \text{for all } \xi \in \mathbf{R}^d. \tag{4}$$

Therefore, it follows fairly readily that when $f \in \mathrm{Dom}[L] \cap L^1(\mathbf{R}^d)$, $Lf \in L^1(\mathbf{R}^d)$, and $\hat{f} \in L^1(\mathbf{R}^d)$, then we have

$$\widehat{Lf}(\xi) = -\Psi(-\xi)\hat{f}(\xi) \qquad \text{for every } \xi \in \mathbf{R}^d. \tag{5}$$

It follows immediately from these calculations that: (i) Every P_t and R_λ map $\mathcal{S}$ to $\mathcal{S}$; and (ii) Therefore, $\mathcal{S}$ is dense in $\mathrm{Dom}[L]$. Therefore, we can try to understand L better by trying to compute Lf not for all $f \in \mathrm{Dom}[L]$, but rather for all f in the dense subcollection $\mathcal{S}$. But the formula for the Fourier transform of Lf [together with the estimate $|\Psi(\xi)| = O(\|\xi\|^2)$] shows that $L : \mathcal{S} \to \mathcal{S}$ and

$$(Lf)(x) = -\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \mathrm{e}^{-i\xi\cdot x}\Psi(-\xi)\hat{f}(\xi)\,\mathrm{d}\xi \qquad \text{for all } x \in \mathbf{R}^d \text{ and } f \in \mathcal{S}.$$

Consider the simplest case that the process X satisfies $X_t = at$ for some $a \in \mathbf{R}^d$; i.e., $\Psi(\xi) = -i(a \cdot \xi)$. In that case, we have

$$\begin{aligned}(Lf)(x) = \frac{1}{(2\pi)^d}\cdot\int_{\mathbf{R}^d}(a\cdot i\xi)\mathrm{e}^{-i\xi\cdot x}\hat{f}(\xi)\,\mathrm{d}\xi &= -\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\mathrm{e}^{-i\xi\cdot x}(a\cdot\widehat{\nabla f}(\xi))\,\mathrm{d}\xi \\ &= -a\cdot(\nabla f)(x),\end{aligned}$$

thanks to the inversion formula. The very same computation works in the more general setting, and yields

Theorem 14. *If $f \in \mathcal{S}$, then $Lf = Cf + Jf$, where*

$$(Cf)(x) = -a\cdot(\nabla f)(x) + \frac{1}{2}\sum\sum_{1\leq i,j\leq d}(\sigma'\sigma)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}f(x),$$

and

$$(Jf)(x) := \int_{\mathbf{R}^d}\left[f(x+z) - f(x) - z\cdot(\nabla f)(x)\mathbb{1}_{[0,1)}(\|z\|)\right]m(\mathrm{d}z) \quad \text{for all } x \in \mathbf{R}^d,$$

Moreover, J is the generator of the pure-jump part; and $C = -a\cdot\nabla + \frac{1}{2}\nabla'\sigma'\sigma\nabla$ is the generator of the continuous/Gaussian part.

Here are some examples:

- If X is Brownian motion on $\mathbf{R}^d$, then $L = \frac{1}{2}\Delta$ is one-half of the Laplace operator [on $\mathcal{S}$];
- If X is the Poisson process on $\mathbf{R}$ with intensity $\lambda \in (0\,,\infty)$, then $(Lf)(x) = \lambda[f(x+1) - f(x)]$ for $f \in \mathcal{S}$ [might be easier to check Fourier transforms];
- If X is the isotropic stable process with index $\alpha \in (0\,,2)$, then for all $f \in \mathcal{S}$,

$$(Lf)(x) = \text{const} \cdot \int_{\mathbf{R}^d} \left[\frac{f(x+z) - f(x) - z \cdot (\nabla f)(x) \mathbb{1}_{[0,1]}(\|z\|)}{\|z\|^{d+\alpha}} \right] \mathrm{d}z.$$

  Since $\widehat{Lf}(\xi) \propto -\hat{f}(\xi) \cdot \|\xi\|^\alpha$, L is called the "fractional Laplacian" with fractional power $\alpha/2$. It is sometimes written as $L = -(-\Delta)^{\alpha/2}$; the notation is justified [and explained] by the symbolic calculus of pseudo-differential operators.

Problems for Lecture 9

1. Prove that $P_t(x\,,A) = P_t(A - x)$ for all $t \geq 0$, $x \in \mathbf{R}^d$, and $A \in \mathcal{B}(\mathbf{R}^d)$, where $A - x := \{a - x : a \in A\}$. Conclude that the Chapman–Kolmogorov equation is equivalent to the following formula for $\mathrm{E} \prod_{j=1}^k f_j(x_0 + X_{t_j})$:

$$\int_{\mathbf{R}^d} P_{t_1}(\mathrm{d}x_1) \int_{\mathbf{R}^d} P_{t_2 - t_1}(\mathrm{d}x_2) \cdots \int_{\mathbf{R}^d} P_{t_k - t_{k-1}}(\mathrm{d}x_k) \prod_{j=1}^k f_j(x_0 + \cdots + x_j),$$

using the same notation as Proposition 2.

2. Suppose $Y \in L^1(\mathrm{P})$ is measurable with respect to $\sigma(\{X_r\}_{r \geq t})$ for a fixed non-random $t \geq 0$. Prove that $\mathrm{E}(Y \mid \mathcal{F}_t) = \mathrm{E}(Y \mid X_t)$ a.s.

3. Verify that $-X := \{-X_t\}_{t \geq 0}$ is a Lévy process; compute its transition measures $\tilde{P}_t(x\,,\mathrm{d}y)$ and verify the following duality relationship: For all measurable $f, g : \mathbf{R}^d \to \mathbf{R}_+$ and $z \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} f(x)\,\mathrm{d}x \int_{\mathbf{R}^d} g(y)\, P_t(x\,,\mathrm{d}y) = \int_{\mathbf{R}^d} g(y)\,\mathrm{d}y \int_{\mathbf{R}^d} f(x)\, \tilde{P}_t(y\,,\mathrm{d}x).$$

4. Prove that $u(s\,,x) := (P_s f)(x)$ solves [weakly] the partial integro-differential equation

$$\frac{\partial u}{\partial s}(s\,,x) = (Lu)(s\,,x) \quad \text{for all } s > 0 \text{ and } x \in \mathbf{R}^d,$$

subject to $u(0\,,x) = f(x)$.

5. Derive the resolvent equation (3).

6. Verify (4) and (5).

7. First, improve Proposition 11 in the case $p = 2$ as follows: Prove that there exists a unique continuous extension of P_t to all of $L^2(\mathbf{R}^d)$. Denote that by P_t still. Next, define

$$\mathrm{Dom}_2[L] := \left\{ f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\Psi(\xi)|^2 \cdot |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}.$$

Then prove that $\lim_{t \downarrow 0} t^{-1}(P_t f - f)$ exists, as a limit in $L^2(\mathbf{R}^d)$, for all $f \in \mathrm{Dom}_2[L]$. Identify the limit when $f \in C_c(\mathbf{R}^d)$.