Lévy Processes

Recall that a *Lévy process* $\{X_t\}_{t\geq 0}$ on $\mathbf{R}^d$ is a cadlag stochastic process on $\mathbf{R}^d$ such that $X_0 = 0$ and X has i.i.d. increments. We say that X is *continuous* if $t \mapsto X_t$ is continuous. On the other hand, X is *pure jump* if $t \mapsto X_t$ can move only when it jumps [this is not a fully rigorous definition, but will be made rigorous en route the Itô–Lévy construction of Lévy processes].

Definition 1. If X is a Lévy process, then its *tail sigma-algebra* is $\mathcal{T} := \cap_{t\geq 0}\sigma(\{X_{r+t} - X_t\}_{r\geq 0})$. □

The following is a continuous-time analogue of the Kolmogorov zero-one law for sequences of i.i.d. random variables.

Proposition 2 (Kolmogorov zero-one law). *The tail sigma algebra of a Lévy process is trivial; i.e.,* $\mathrm{P}(A) \in \{0\,,1\}$ *for all* $A \in \mathcal{T}$.

The Lévy–Itô construction

The following is the starting point of the classification of Lévy processes, and is also known as the *Lévy–Khintchine formula*; compare with the other Lévy–Khintchine formula (Theorem 6).

Theorem 3 (The Lévy–Khintchine formula; Itô, 1942; Lévy, 1934). *For every Lévy exponent* Ψ *on* $\mathbf{R}^d$ *there exists a Lévy process* X *such that for all* $t \geq 0$ *and* $\xi \in \mathbf{R}^d$,

$$\mathrm{E}e^{i\xi\cdot X_t} = e^{-t\Psi(\xi)}. \tag{1}$$

Conversely, if X *is a Lévy process on* $\mathbf{R}^d$ *then* (1) *is valid for a Lévy exponent* Ψ.

In words, the collection of all Lévy processes on $\mathbf{R}^d$ is in one-to-one correspondence with the family of all infinitely-divisible laws on $\mathbf{R}^d$.

We saw already that if X is a Lévy process, then X_1 [in fact, X_t for every $t \ge 0$] is infinitely divisible. Therefore, it remains to prove that if Ψ is a Lévy exponent, then there is a Lévy process X whose exponent is Ψ. The proof follows the treatment of Itô (1942), and is divided into two parts.

Isolating the pure-jump part. Let $B := \{B_t\}_{t\ge 0}$ be a d-dimensional Brownian motion, and consider the Gaussian process defined by

$$W_t := \sigma B_t - at. \qquad (t \ge 0).$$

A direct computation shows that $W := \{W_t\}_{t\ge 0}$ is a continuous Lévy process with Lévy exponent

$$\Psi^{(c)}(\xi) = ia'\xi + \frac{1}{2}\|\sigma\xi\|^2 \qquad \text{for all } \xi \in \mathbf{R}^d.$$

[W is a Brownian motion with drift $-a$, where the coordinates of W are possibly correlated, unless σ is diagonal.] Therefore, it suffices to prove the following:

Proposition 4. *There exists a pure-jump Lévy process Z with characteristic exponent*

$$\Psi^{(d)}(\xi) := \int_{\mathbf{R}^d} \left(1 - e^{i\xi\cdot z} + i(\xi\cdot z)\mathbb{1}_{(0,1)}(\|z\|)\right) m(\mathrm{d}z),$$

for all $\xi \in \mathbf{R}^d$.

Indeed, if this were so, then we could construct W and Z independently from one another, and set

$$X_t = W_t + Z_t \qquad \text{for all } t \ge 0.$$

This proves Theorem 3, since $\Psi = \Psi^{(c)} + \Psi^{(d)}$. In fact, together with Theorem 6, this implies the following:

Theorem 5. *(1) The only continuous Lévy processes are Brownian motions with drift, and; (2) The continuous [i.e., Gaussian] and pure-jump parts of an arbitrary Lévy process are independent from one another.*

Therefore, it suffices to prove Proposition 4.

Proof of Proposition 4. Consider the measurable sets

$$A_{-1} := \left\{z \in \mathbf{R}^d : \ \|z\| \ge 1\right\}, \quad \text{and} \quad A_n := \left\{z \in \mathbf{R}^d : 2^{-n+1} \le \|z\| < 2^{-n}\right\},$$

as n varies over all nonnegative integers. Now we can define stochastic processes $\{X^{(n)}\}_{n=-1}^{\infty}$ as follows: For all $t \geq 0$,

$$X_t^{(-1)} := \int_{A_{-1}} x\,\Pi_t(\mathrm{d}x), \qquad X_t^{(n)} := \int_{A_n} x\,\Pi_t(\mathrm{d}x) - tm(A_n) \qquad (n \geq 0).$$

Thanks to the construction of Lecture 5 (pp. 26 and on), $\{X^{(n)}\}_{n=-1}^{\infty}$ are independent Lévy processes, and for all $n \geq 0$, $t \geq 0$, and $\xi \in \mathbf{R}^d$,

$$\mathrm{E}e^{i\xi\cdot X_t^{(n)}} = \exp\left\{-t\int_{A_n}\left(1 - \mathrm{e}^{i\xi\cdot z} + i(\xi\cdot z)\mathbb{1}_{(0,1)}(\|z\|)\right) m(\mathrm{d}z)\right\}.$$

Moreover, $X^{(-1)}$ is a compound Poisson process with parameters $m(\bullet \cap A_{-1})/m(A_{-1})$ and $\lambda = m(A_{-1})$, for all $n \geq 0$, $X^{(n)}$ is a compensated compound Poisson process with parameters $m(\bullet \cap A_n)/m(A_n)$ and $\lambda = m(A_n)$.

Now $Y_t^{(n)} := \sum_{k=0}^{n} X_t^{(k)}$ defines a Lévy process with exponent

$$\psi_n(\xi) := \int_{1>\|z\|\geq 2^{-n+1}}\left(1 - \mathrm{e}^{i\xi\cdot z} + i(\xi\cdot z)\mathbb{1}_{(0,1)}(\|z\|)\right) m(\mathrm{d}z),$$

valid for all $\xi \in \mathbf{R}^d$ and $n \geq 1$. Our goal is to prove that there exists a process Y such that for all nonrandom $T > 0$,

$$\sup_{t\in[0,T]}\left\|Y_t^{(n)} - Y_t\right\| \to 0 \qquad \text{in } L^2(\mathrm{P}). \tag{2}$$

Because $Y^{(n)}$ is cadlag for all n, uniform convergence shows that Y is cadlag for all n. In fact, the jumps of $Y^{(n+1)}$ contain those of $Y^{(n)}$, and this proves that Y is pure jump. And because the finite-dimensional distributions of $Y^{(n)}$ converge to those of Y, it follows then that Y is a Lévy process, independent of $X^{(-1)}$, and with characteristic exponent

$$\psi_\infty(\xi) = \lim_{n\to\infty}\psi_n(\xi) = \int_{1>\|z\|}\left(1 - \mathrm{e}^{i\xi\cdot z} + i(\xi\cdot z)\mathbb{1}_{(0,1)}(\|z\|)\right) m(\mathrm{d}z).$$

[The formula for the limit holds by the dominated convergence theorem.] Sums of independent Lévy processes are themselves Lévy. And their exponents add. Therefore, $X_t^{(-1)} + Y_t$ is Lévy with exponent $\Psi^{(d)}$.

It remains to prove the existence of Y. Let us choose and fix some $T > 0$, and note that for all $j, k \geq 1$ and $t \geq 0$,

$$Y_t^{(n+k)} - Y_t^{(n)} = \sum_{j=k+1}^{n+k}\left(\int_{A_j} x\,\Pi_t(\mathrm{d}x) - tm(A_j)\right),$$

and the summands are independent because the A_j's are disjoint. Since the left-hand side has mean zero, it follows that

$$\begin{aligned} \mathrm{E}\left(\left\| Y_t^{(n+k)} - Y_t^{(n)} \right\|^2\right) &= \sum_{j=k+1}^{n+k} \mathrm{E}\left(\left\| \int_{A_j} x\, \Pi_t(\mathrm{d}x) - tm(A_j) \right\|^2\right) \\ &\le 2^{d-1} t \sum_{j=k+1}^{n+k} \int_{A_j} \|x\|^2\, m(\mathrm{d}x) = 2^{d-1} t \int_{\cup_{j=k+1}^{n+k} A_j} \|x\|^2\, m(\mathrm{d}x); \end{aligned}$$

see Theorem 3. Every one-dimensional mean-zero Lévy process is a mean-zero martingale [in the case of Brownian motion we have seen this in Math. 6040; the reasoning in the general case is exactly the same]. Therefore, $Y^{(n+k)} - Y^{(n)}$ is a mean-zero cadlag martingale (coordinatewise). Doob's maximal inequality tells us that

$$\mathrm{E}\left(\sup_{t\in[0,T]} \left\| Y_t^{(n+k)} - Y_t^{(n)} \right\|^2\right) \le 2^{d+1} T \int_{2^{-k} \le \|z\| < 2^{n-k+1}} \|x\|^2\, m(\mathrm{d}x).$$

This and the definition of a Lévy measure (p. 3) together imply (2), whence the result. □

Problems for Lecture 6

1. Prove the Kolmogorov 0-1 law (page 29).

2. Prove that every Lévy process X on $\mathbf{R}^d$ is a strong Markov process. That is, for all finite stopping times T [in the natural filtration of X], $t_1, \ldots, t_k \ge 0$, and $A_1, \ldots, A_k \in \mathscr{B}(\mathbf{R}^d)$,

$$\mathrm{P}\left(\bigcap_{j=1}^{k} \left\{ X_{T+t_j} - X_T \in A_j \right\} \,\middle|\, \mathscr{F}_T \right) = \mathrm{P}\left(\bigcap_{j=1}^{k} \left\{ X_{t_j} \in A_j \right\}\right) \quad \text{a.s.}$$

(Hint: Follow the Math. 6040 proof of the strong Markov property of Brownian motion.)