Chapter 8

Martingales

8.18. Because $\mathcal{F}_{k+n-1} \subset \mathcal{F}_n$ for all $k, n \geqslant 1$, we may apply the tower property of conditional expectations to see that

$$E[X_{k+n} \mid \mathcal{F}_n] = E\left(E[X_{k+n} \mid \mathcal{F}_{k+n-1}] \,\middle|\, \mathcal{F}_n\right) \geqslant E[X_{k+n-1} \mid \mathcal{F}_n],$$

almost surely. Now apply induction to deduce the following: A.s.,

$$E[X_{k+n} \mid \mathcal{F}_n] \geqslant E[X_{k+n-1} \mid \mathcal{F}_n] \geqslant \cdots \geqslant E[X_n \mid \mathcal{F}_n] = X_n.$$

8.19. Suppose we could write our submartingale X as: $X_n = M_n + Z_n$ and $X_n = M'_n + Z'_n$, where M and M' are martingales, and Z and Z' are previsible increasing processes. Because $M - M'$ defines a martingale, this proves that $Z_n - Z'_n$ defines a martingale also. That is,

$$E\left[Z_n - Z'_n \mid \mathcal{F}_{n-1}\right] = Z_{n-1} - Z'_{n-1} \qquad \text{a.s.}$$

But $Z - Z'$ is previsible. So the preceding also equals $Z_n - Z'_n$ a.s. This proves that for all $n \geqslant 1$, $Z_n - Z_{n-1} = Z'_n - Z'_{n-1}$ a.s. Thus, for all $m \geqslant 1$, $Z_m = Z_1 + \sum_{j=2}^{m}(Z_j - Z_{j-1}) = Z_1 + \sum_{j=2}^{m}(Z'_j - Z'_{j-1}) = Z_1 - Z'_1 + Z'_m$ a.s. Thus, if we insist that $Z_1 = Z'_1 = E[X_1]$, as was the case in the proof of Theorem 8.20, then $Z_m = Z'_m$ a.s. for all m.

8.20. If X is bounded in $L^1(P)$ then $E[X_n^+] \leqslant E[X_n^+] + E[X_n^-] = \|X_n\|_1$ is bounded. For the converse note that $E[X_n^-] = E[X_n^+] - E[X_n] = E[X_n^+] - E[X_1]$ by the martingale property. This proves that $\|X_n\|_1 = 2E[X_n^+] - E[X_1]$ is bounded also.

8.26. Without loss of generality, we assume that $Y \geqslant 0$; otherwise, we consider Y^+ and Y^- separately.

Let $\mathcal{F}_n := \sigma(X_n)$ and note that $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a filtration of σ-algebras in $\mathcal{F}$. [This is because of the nested structure of dyadic rationals.] By the martingale convergence theorem, $Z := \lim_{n\to\infty} \mathrm{E}(Y\,|\,X_n)$ exists a.s. and is finite a.s. Our goal is to prove that $Z = \mathrm{E}(Y\,|\,X)$.

Let I be a dyadic-rational interval, and note that if the length $|I|$ is at least 2^{-N}, then $\{X \in I\} = \{X_n \in I\}$ for all $n \geqslant N$. Therefore,

$$\begin{aligned}
\mathrm{E}\left[\mathrm{E}(Y\,|\,X_n);\ X \in I\right] &= \mathrm{E}\left[\mathrm{E}(Y\,|\,X_n);\ X_n \in I\right] \\
&= \mathrm{E}[Y;\ X_n \in I] \\
&= \mathrm{E}[Y;\ X \in I] \\
&= \mathrm{E}\left[\mathrm{E}(Y\,|\,X);\ X \in I\right] \qquad \text{for all } n \geqslant N.
\end{aligned}$$

Recall that $Z := \lim_{n\to\infty} \mathrm{E}(Y\,|\,X_n)$ exists a.s. and is finite a.s. If we knew, additionally, that Y is a bounded random variable—say $|Y| \leqslant K$ a.s. for a constant K—then $|\mathrm{E}(Y\,|\,X_n)| \leqslant \mathrm{E}(|Y|\,|\,X_n) \leqslant K$ by the conditional Jensen inequality. Consequently, the bounded convergence theorem would yield

$$\mathrm{E}\left[Z;\ X \in I\right] = \mathrm{E}\left[\mathrm{E}(Y\,|\,X);\ X \in I\right],$$

for all dyadic-rational intervals I. Since dyadic rational intervals generate $\mathcal{B}(\mathbf{R})$, it follows that we could in fact choose any Borel set I in the preceding display, and therefore, it follows that $\mathrm{E}[Z; A] = \mathrm{E}[\mathrm{E}(Y\,|\,X); A]$ for all $A \in \sigma(X)$. Since Z and $\mathrm{E}(Y\,|\,X)$ are both $\sigma(X)$-measurable, the uniqueness of conditional expectations yields $Z = \mathrm{E}(Y\,|\,X)$, as desired.

When Y is not bounded we can find a bounded random variable Y_K such that $\|Y - Y_K\|_1 \leqslant K^{-1}$ for all constants $K \geqslant 1$. The preceding shows that $\lim_{n\to\infty} \mathrm{E}(Y_K\,|\,X_n) = \mathrm{E}(Y_K\,|\,X)$ a.s. This convergence holds also in $L^1(P)$ because of the bounded convergence theorem.

But now we note from conditional Jensen inequality that

$$\|\mathrm{E}(Y\,|\,X_n) - \mathrm{E}(Y_K\,|\,X_n)\|_1 \leqslant K^{-1},$$

and

$$\|\mathrm{E}(Y\,|\,X) - \mathrm{E}(Y_K\,|\,X)\|_1 \leqslant K^{-1}.$$

Therefore,

$$\|\mathrm{E}(Y\,|\,X_n) - \mathrm{E}(Y\,|\,X)\|_1 \leqslant 2K^{-1} + \|\mathrm{E}(Y_K\,|\,X_n) - \mathrm{E}(Y_K\,|\,X)\|_1 \to 2K^{-1},$$

as $n \to \infty$. Let $K \to \infty$ to conclude that $\mathrm{E}(Y\,|\,X_n) \to \mathrm{E}(Y\,|\,X)$ in $L^1(P)$. Since $\mathrm{E}(Y\,|\,X_n) \to Z$ a.s., it follows that $Z = \mathrm{E}(Y\,|\,X)$ a.s.

Note that we have proved, for free, the additional fact that $\mathrm{E}(Y\,|\,X_n) \to \mathrm{E}(Y\,|\,X)$ in $L^1(P)$.