Chapter 4

Integration

4.1. There are two things to prove here: (i) $\sigma(X)$ is σ-algebra; and (ii) it is the smallest one with respect to which X is measurable.

As regards (i), we check that $\varnothing \in \sigma(X)$ because $X^{-1}(\varnothing) = \varnothing$. Also if $A \in \sigma(X)$ then $A = X^{-1}(B)$ for some $B \in \mathcal{A}$. But then $A^c = (X^{-1}(B))^c$, which is in $\sigma(X)$. Finally, suppose $A_1, A_2, \ldots$ are all in $\sigma(X)$. Then we can find $B_1, B_2, \ldots$ such that $A_i = X^{-1}(B_i)$. Evidently, $\cup_{i=1}^{\infty} X^{-1}(B_i) = X^{-1}(\cup_{i=1}^{\infty} B_i)$. Because $\cup_{i=1}^{\infty} B_i \in \mathcal{A}$ (the latter is after all a σ-algebra), it follows that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} X^{-1}(B_i) \in \sigma(X)$. We have proved that $\sigma(X)$ is a σ-algebra.

Note that X is measurable with respect to a σ-algebra $\mathcal{G}$ iff $X^{-1}(B) \in \mathcal{G}$ for all $B \in \mathcal{A}$. Therefore, a priori, X is measurable with respect to $\sigma(X)$, and any other $\mathcal{G}$ must contain $\sigma(X)$.

4.2. Define $I(j, n) := [j2^{-n}, (j+1)2^{-n})$, and set

$$\bar{f}_n(\omega) := \sum_{j \in \mathbf{Z}} \left(\frac{j+1}{2^n}\right) \mathbf{1}_{I(j,n)}(f(\omega)), \quad \underline{f}_n(\omega) := \sum_{j \in \mathbf{Z}} \left(\frac{j}{2^n}\right) \mathbf{I}_{I(j,n)}(f(\omega)).$$

Because f is bounded, these are finite sums. Also, the measurability of f ensures that $\underline{f}_n$ and $\bar{f}_n$ are measurable. Finally, note that $\underline{f}_n(\omega) \leqslant f(\omega) \leqslant \bar{f}_n(\omega)$. Also, $\bar{f}_n(\omega) \geqslant \bar{f}_{n+1}(\omega)$, whereas $\underline{f}_n(\omega) \leqslant \underline{f}_{n+1}(\omega)$. This does the job.

4.3. By concentrating on f^+ and then f^- separately, we may assume without loss of generality that $f \geqslant 0$. Consider the proposed identity:

$$\int f \, d\mu = \sum_{x \in \Omega} f(x). \tag{4.1}$$

This holds, by definition, if $f(x) = \mathbf{1}_A(x)$ for any $A \subseteq \Omega$. [The summability of f ensures that Ω is at most countable, so measurability issues do not arise.] Therefore, (1.1) is valid for all elementary functions f. Choose a sequence of elementary functions f_n converging up to f pointwise. Eq. (1.1) then follows from the monotone convergence theorem.

4.4. If $A_1, A_2, \ldots$ are disjoint and measurable then so are $f^{-1}(A_1), f^{-1}(A_2), \ldots$, and $f^{-1}(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} f^{-1}(A_n)$. The rest is easy sailing.