3.3. For this exercise, we need Cantor's countable axiom of choice: *A countable union of countable sets is itself countable.*

Let $\mathscr{F}$ denote the σ-algebra generated by all singletons in $\mathbf{R}$. Obviously, $\{x\} \in \mathscr{F}$ for all $x \in \mathbf{R}$. It remains to show that $[a,b] \notin \mathscr{F}$ for any $a \leq b$. Define $\mathscr{G} = \{G \subseteq \mathbf{R} : \text{ either } G \text{ or } G^c \text{ is denumerable}\}$. [Recall that "denumerable" means "at most countable."] You can check directly that $\mathscr{G}$ is a σ-algebra. We claim that $\mathscr{F} = \mathscr{G}$. This would prove that $[a,b] \notin \mathscr{F}$, because neither $[a,b]$ nor its complement are denumerable.

Let $\mathscr{A}$ denote the algebra generated by all singletons. If $E \in \mathscr{A}$, then either E is a finite collection of singletons, or E^c is. This proves that $\mathscr{A} \subseteq \mathscr{G}$. The monotone class theorem proves that $\sigma(\mathscr{A}) \subseteq \mathscr{G}$, but $\sigma(\mathscr{A}) = \mathscr{F}$. Therefore, it remains to prove that $\mathscr{G} \subseteq \mathscr{F}$. But it is manifest that $\mathscr{G} \subset \sigma(\mathscr{A})$.