Midterm Exam, Math 6010-1

Fall 2016
September 28, 2016

> This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours.
>
> The exam is made of 3 questions in 4 pages, and is worth 40 points, total. Be sure to try all of the problems.
>
> Partial credit is given only to carefully-written solutions.

1. (15 points in 3 equal parts) Which of the following are quadratic forms? Explain your reasoning, and identify the matrix of each quadratic form explicitly.

   (a) (5 points) $Q(\boldsymbol{x}) = \sum_{i=1}^{n} x_i$ for every $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$.

   Solution. This is not a quadratic form. Here is why. Suppose to the contrary that there existed a matrix $\boldsymbol{A}$ such that $Q(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$. That is, suppose

   $$Q(\boldsymbol{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j} x_i x_j.$$

   Then, $(\partial^2 Q(\boldsymbol{x}))/(\partial x_i \partial x_j) = A_{i,j}$ for every $1 \le i, j \le n$. For us, however, $(\partial^2 Q(\boldsymbol{x}))/(\partial x_i \partial x_j) = 0$ for every i, j. This implies that $A_{i,j} = 0$ for all i, j, which means that $\boldsymbol{A} = \boldsymbol{0}$, an impossibility.

(b) (5 points) $Q(\boldsymbol{x}) = \sum_{i=1}^n \sum_{j=1}^n (x_i - \bar{x})(x_j - \bar{x})$ for every $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$, where $\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i$.

Solution. First, notice that $Q(\boldsymbol{x}) = \sum_{i=1}^n (x_i - \bar{x}) \cdot \sum_{j=1}^n (x_j - \bar{x}) = 0$. Thus, we see that Q is a quadratic form with trivial matrix $\boldsymbol{A} = \mathbf{0}$.

(c) (5 points) $Q(\boldsymbol{x}) = \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$ for every $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$.

Solution. Write $Q = Q_1 + Q_2 - 2Q_3$, where

$$Q_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} x_{i+1}^2, \quad Q_2(\boldsymbol{x}) = \sum_{i=1}^{n-1} x_i^2, \quad Q_3(\boldsymbol{x}) = \sum_{i=1}^{n-1} x_i x_{i+1}.$$

Each Q_j is a quadratic form, with respective matrices,

$$\boldsymbol{A}_1 = \begin{pmatrix} \mathbf{0}_{1\times(n-1)} & 0 \\ \boldsymbol{I}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} \boldsymbol{I}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} \\ \mathbf{0}_{1\times(n-1)} & 0 \end{pmatrix},$$

and $\boldsymbol{A}_3$ is a matrix that is all ones but the super-diagonal entire; those super-diagonal entries are all ones; that is,

$$\boldsymbol{A}_3 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, Q is a quadratic form with matrix $\boldsymbol{A}_1 - 2\boldsymbol{A}_3 + \boldsymbol{A}_2$.

2. (10 points total) The claim has been made that a certain random vector $\boldsymbol{X}$ has a $\mathrm{N}_3(\mathbf{0}\,,\boldsymbol{\Sigma})$ distribution, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 10 & 2 & 3 \\ 4 & 200 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(a) (2 points) The eigenvalues of $\boldsymbol{\Sigma}$ are (approximately) 4.83, 13.87, and 200.3. Can the claim possibly be true?

Solution. It can potentially be true because $\boldsymbol{\Sigma}$ is positive semi-definite.

(b) (5 points) Compute $\mathrm{E}(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})$ for the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution. We know that

$$\mathrm{E}(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}) = \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu} + \mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}\Sigma_{i,j} = \sum_{i=1}^{3} A_{i,i}\Sigma_{i,i} = 20 - 200 + 18 = -162.$$

(c) (3 points) Explain why your answer to (b) implies that $\boldsymbol{A}$ is not positive semi-definite.

Solution. If $\boldsymbol{A}$ were positive semidefinite, then $\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ would have to be ≥ 0, and that would imply that $\mathrm{E}(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})$ would have to be ≥ 0.

3. (15 points total) Let $X_1, \ldots, X_{10}$ be an independent random sample from a $\mathrm{N}(0\,,1)$ distribution, and define $\boldsymbol{Y} = (Y_1\,, Y_2)'$ as the vector of the following two "lagged averages":

$$Y_1 = \frac{X_1 + \cdots + X_5}{5} \quad \text{and} \quad Y_2 = \frac{X_2 + \cdots + X_6}{5}.$$

(a) (10 points) Compute $\mathrm{E}(\boldsymbol{Y})$ and $\mathrm{Var}(\boldsymbol{Y})$.

Solution. $\mathrm{E}(Y_1) = \mathrm{E}(Y_2) = 0$, therefore, $\mathrm{E}(\boldsymbol{Y}) = \mathbf{0}$. Also, $\mathrm{Var}(Y_1) = \mathrm{Var}(Y_2) = {}^1/_5$, and $\mathrm{Cov}(Y_1\,, Y_2) = \frac{1}{36}\mathrm{Var}(X_2 + \cdots + X_5) = {}^4/_{36} = {}^1/_8$. Therefore,

$$\mathrm{Var}(\boldsymbol{Y}) = \begin{bmatrix} {}^1/_5 & {}^1/_8 \\ {}^1/_8 & {}^1/_5 \end{bmatrix}.$$

(b) (5 points) Explain carefully why $\boldsymbol{Y}$ has a bivariate normal distribution. Does $\boldsymbol{Y}$ have a probability density?

Solution. $\boldsymbol{Y}$ is a bivariate normal because $\boldsymbol{Z} = (X_1\,, \ldots, X_6)'$ is a vector of i.i.d. standard normals, and

$$\boldsymbol{Y} = \begin{pmatrix} {}^1/_5 & {}^1/_5 & {}^1/_5 & {}^1/_5 & {}^1/_5 & 0 \\ 0 & {}^1/_5 & {}^1/_5 & {}^1/_5 & {}^1/_5 & {}^1/_5 \end{pmatrix} \boldsymbol{Z}.$$

The determinant of $\mathrm{Var}(\boldsymbol{Y})$ is positive, so $\mathrm{Var}(\boldsymbol{Y})$ is positive definite and hence $\boldsymbol{Y}$ has a pdf. Indeed,

$$\det\left[\mathrm{Var}(\boldsymbol{Y})\right] = ({}^1/_{25}) - ({}^1/_{64}) = {}^{39}/_{(25\times 64)} > 0.$$