Math 5080–1
Solutions to homework 3

June 12, 2012

 # 1. Stated in technical terms, the question is this: Let $X_1, \ldots, X_{20}$ be independent with common distribution $\text{N}(101\,,4)$. Thus, for example $E(X_1) = 101$ pounds. What is $P\{X_1 + \cdots + X_{20} > 2000\}$? [1 ton = 2,000 pounds]. Since $X_1 + \cdots + X_{20} \sim \text{N}(2020\,,80)$, the probability that we seek is

$$P\left\{\text{N}(2020\,,80) > 2000\right\} = P\left\{\text{N}(0\,,1) > \frac{2000-2020}{\sqrt{80}}\right\} \approx 1 - \Phi(-2.23).$$

We wish to use Table 3 (p. 603): $1 - \Phi(-2.23) = \Phi(2.23) \approx 0.9871$.

 # 2. (a) We are asked to compute $P\{S > B\} = P\{S - B > 0\}$. Since $S - B \sim \text{N}(-0.01, 0.0013)$,

$$P\{S > B\} = P\left\{\text{N}(-0.01, 0.0013) > 0\right\} = P\left\{\text{N}(0\,,1) > \frac{0.01}{\sqrt{0.0013}}\right\} \approx P\{\text{N}(0\,,1) > 0.28\}.$$

That is, we want $1 - \Phi(0.28) \approx 1 - 0.6103 = 0.3897$.

(b) "Non interference" means "the selected shaft has a smaller diameter than the selected bearing." Suppose $S \sim \text{N}(1\,,\sigma^2)$ and $B \sim \text{N}(1.01\,,\sigma^2)$ are independent. We are asked to find σ such that $P\{S < B\} = 0.95$. Since $S - B \sim \text{N}(-0.01\,,2\sigma^2)$, we are led to

$$0.95 = P\left\{\text{N}(-0.01\,,2\sigma^2) < 0\right\} = P\left\{\text{N}(0\,,1) < \frac{0.01}{\sqrt{2}\,\sigma}\right\} = \Phi\left(\frac{0.01}{\sqrt{2}\,\sigma}\right).$$

The normal table (Table 3) shows that $\Phi(1.65) \approx 0.9505$. Therefore, we set

$$\frac{0.01}{\sqrt{2}\,\sigma} \approx 1.65 \;\Rightarrow\; \sigma \approx \frac{0.01}{1.65 \times \sqrt{2}} \approx 0.004285.$$

 # 3. (a) $\bar{X}$ is unbiased for μ $[E(\bar{X}) = \mu]$ and S^2 is unbiased for σ^2 $[E(S^2) = \sigma^2]$. Therefore, $2\bar{X} - 5S^2$ is unbiased for $\theta = 2\mu - 5\sigma^2$. Now we write this statistic in terms of U and W.

The first is easy: $\bar{X} = U/n$; the second is obtained by recalling that

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}(\bar{X})^2 = \frac{W}{n-1} - \frac{U^2}{n(n-1)}.$$

Therefore, an unbiased estimator of θ, in terms of U and W, is

$$\frac{2U}{n} - 5\left[\frac{W}{n-1} - \frac{U^2}{n(n-1)}\right].$$

(b) $E(X_1^2) = \sigma^2 + \mu^2$. Therefore, $E(W/n) = \sigma^2 + \mu^2$ also. This means that W/n is unbiased for $\sigma^2 + \mu^2$.

(c) In terms of indicator functions, $Y_i = I\{X_i \le c\}$. Therefore, $E(Y_i) = P\{X_i \le c\} = F_X(c) = \Phi((c-\mu)/\sigma)$. This shows that each Y_i is an unbiased estimator of $F_X(c)$. Better still, so is $n^{-1}\sum_{i=1}^{n} Y_i$.

 # 4. This is an exercise in changing variables. First, we backsolve:

$$X_1 = \frac{Y_1 + Y_2}{2}, \; X_2 = \frac{Y_1 - Y_2}{2}.$$

The Jacobian is

$$J = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}.$$

Therefore,

$$f_{\boldsymbol{Y}}(y_1\,, y_2) = f_{\boldsymbol{X}}\left(\frac{y_1+y_2}{2}\,, \frac{y_1-y_2}{2}\right) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{y_1+y_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2\right]\right).$$

But

$$\left(\frac{y_1+y_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 = \frac{y_1^2}{2} + \frac{y_2^2}{2}.$$

Therefore,

$$f_{\boldsymbol{Y}}(y_1\,, y_2) = f_{\boldsymbol{X}}\left(\frac{y_1+y_2}{2}\,, \frac{y_1-y_2}{2}\right) = f_{X_1}(y_1) f_{X_2}(y_2).$$

It follows that Y_1 and Y_2 are independent $\mathrm{N}(\mu\,, \sigma^2)$'s.

 # 5. (a) $T_1 + \cdots + T_{10} \sim \mathrm{Gamma}(100\,, 10)$.

(b) One and one-half years is 547.5 days. Therefore, we are asked to compute $P\{\sum_{i=1}^{10} T_i > 547.5\}$. Recall that $(2/\theta)\mathrm{Gamma}(\theta\,, \kappa) = \chi^2(2\kappa)$. Apply this with $\theta = 100$ and $\kappa = 10$ to obtain

$$P\left\{\sum_{i=1}^{100} T_i > 547.5\right\} = P\left\{\chi^2(20) > \frac{547.5}{50}\right\} = P\left\{\chi^2(20) > 10.95\right\}.$$

According to Table 5 on page 606, $P\{\chi^2(20) < 11\} \approx 0.054$. Therefore,

$$P\left\{\sum_{i=1}^{100} T_i > 547.5\right\} \approx 1 - 0.054 \approx 0.946.$$

(c) Two years is 730 days; therefore, we are asked to find a number N such that

$$P\left\{\sum_{i=1}^{N} T_i > 730\right\} \approx 0.95.$$

We relate this to a χ^2 distribution, as before:

$$P\left\{\sum_{i=1}^{N} T_i > 730\right\} = P\left\{\text{Gamma}(100\,, N) > 730\right\} = P\left\{\chi^2(2N) > \frac{730}{50}\right\} = P\left\{\chi^2(2N) > 14.6\right\}.$$

In other words, we seek to find N such that $P\{\chi^2(2N) \le 14.6\} \approx 0.05$. Table 5 (p. 606) of your text shows that $P\{\chi^2(24) < 14\} \approx 0.053$. Therefore, $2N > 24$, whence any $N > 12$ ought to work.

 # 6. (a) $T_1 + \cdots + T_{10} \sim \text{Gamma}(100\,, 12)$.

(b) As in #5,

$$P\left\{\sum_{i=1}^{10} T_i > 547.5\right\} = P\left\{\chi^2(24) > 10.95\right\} \approx 1 - 0.011 \approx 0.99.$$

(c) As in #5,

$$P\left\{\sum_{i=1}^{N} T_i > 730\right\} = P\left\{\text{Gamma}(100\,, 1.2N) > 730\right\} = P\left\{\chi^2(2.4N) > 14.6\right\}.$$

We know that $0.05 \approx P\{\chi^2(2.4N) \le 14.6\}$; table 5 (p. 606) shows that $P\{\chi^2(24) < 14\} \approx 0.053$; therefore, $2.4N > 24$ works; i.e., any $N > 10$ will do.