The geometric distribution

So far, we have seen only examples of random variables that have a finite number of possible values. However, our rules of probability allow us to also study random variables that have a countable [but possibly infinite] number of possible values. The word "countable" means that you can label the possible values as 1,2,.... A theorem of Cantor states that the number of elements of the real line is uncountable. And so is the number of elements of any nonempty closed/open interval. Therefore, the condition that a random variable X has a countable number of possible values is a restriction.

We say that X has the *geometric distribution* with parameter $p := 1-q$ if

$$P\{X = i\} = q^{i-1}p \qquad (i = 1, 2, \ldots).$$

We have seen this distribution before. Here is one way it arises naturally: Suppose we toss a p-coin [i.e., $P(\text{heads}) = p$, $P(\text{tails}) = q = 1-p$] until the first heads arrives. If X denotes the number of tosses, then X has the Geometric(p) distribution.

Example 1. Suppose X has the Geometric(p) distribution. Then

$$P\{X \le 3\} = P\{X = 1\} + P\{X = 2\} + P\{X = 3\} = p + pq + pq^2.$$

Here is an alternative expression for $P\{X \le 3\}$:

$$P\{X \le 3\} = 1 - P\{X \ge 4\} = 1 - \sum_{i=4}^{\infty} q^{i-1}p = 1 - p\sum_{j=3}^{\infty} q^j.$$

Simple facts about geometric series then tell us that

$$P\{X \le 3\} = 1 - p\left(\frac{q^3}{1-q}\right) = 1 - q^3,$$

since $p = 1 - q$. More generally, if k is a positive integer ≥ 1, then

$$P\{X \leq k\} = \sum_{i=1}^{k} q^{i-1} p = p \sum_{i=1}^{k} q^{i-1} = p \sum_{j=0}^{k-1} q^j = p \left(\frac{1 - q^k}{1 - q} \right) = 1 - q^k,$$

since $p = 1 - q$.

Example 2. What is $E(X)$ when X has the Geometric(p) distribution? Well, clearly,

$$E(X) = \sum_{i=1}^{\infty} i q^{i-1} p = p \sum_{i=1}^{\infty} i q^{i-1}.$$

How can we calculate this? Note that $iq^{i-1} = \frac{d}{dq} q^i$. Therefore a standard fact from calculus [term-by-term differentiation of an infinite series] tells us that

$$E(X) = p \frac{d}{dq} \sum_{i=1}^{\infty} q^i = p \frac{d}{dq} \sum_{i=0}^{\infty} q^i = p \frac{d}{dq} \left(\frac{1}{1 - q} \right) = p \left(\frac{1}{(1 - q)^2} \right) = \frac{1}{p}.$$

Note that we benefited greatly from studying the problem for a general variable p [as opposed to $p = 1/2$, say, or some such choice]. In this way, we were able to use the rules of calculus in order to obtain the answer.

Example 3. What is Var(X) when X has the Geometric(p) distribution? We know that $E(X) = 1/p$. Therefore, it suffices to compute $E(X^2)$. But

$$E(X^2) = \sum_{i=1}^{\infty} i q^{i-1} p = p \sum_{i=1}^{\infty} i^2 q^{i-1}.$$

In order to compute this we can write the sum as

$$\sum_{i=1}^{\infty} i^2 q^{i-1} = \sum_{i=1}^{\infty} i(i-1) q^{i-1} + \sum_{i=1}^{\infty} i q^{i-1}.$$

We saw earlier that

$$\sum_{i=1}^{\infty} i q^{i-1} = \frac{d}{dq} \sum_{i=0}^{\infty} q^i = \frac{1}{p^2}.$$

Similarly,

$$\sum_{i=1}^{\infty} i(i-1) q^{i-1} = q \sum_{i=1}^{\infty} i(i-1) q^{i-2} = q \frac{d^2}{dq^2} \sum_{i=0}^{\infty} q^i = \frac{2q}{p^3}.$$

Therefore, $\sum_{i=1}^{\infty} i^2 q^{i-1} = 2qp^{-3} + p^{-2}$, and hence

$$E(X^2) = \frac{2q}{p^2} + \frac{1}{p} = \frac{2 - p}{p^2} \quad \Rightarrow \quad \text{Var}(X) = \frac{2 - p}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Equivalently, SD$(X) = \sqrt{q}/p$.

The negative binomial distribution

The negative binomial distribution is a generalization of the geometric [and *not* the binomial, as the name might suggest]. Let us fix an integer $r \geq 1$; then we toss a p-coin until the rth heads occur. Let X_r denote the total number of tosses.

Example 4 (The negative binomial distribution)**.** What is the distribution of X_r? First of all, the possible values of X_r are $r, r+1, r+2, \ldots$. Now for all integers $n \geq r$,

$$P\{X_r = n\} = P\left\{\text{The first } n-1 \text{ tosses have } r-1 \text{ heads, and the } n\text{th toss is heads}\right\}.$$

Therefore,

$$P\{X_r = n\} = \binom{n-1}{r-1} p^{r-1} q^{(n-1)-(r-1)} \times p = \binom{n-1}{r-1} p^r q^{n-r} \qquad (n \geq r).$$

This looks a little bit like a binomial probability, except the variable of the mass function is n. This distribution is called the *negative binomial distribution* with parameters $(r\,, p)$.

Example 5. Compute $E(X_r)$ and $\mathrm{Var}(X_r)$. Let G_1 denote the number of tosses required to obtain the first heads. Then define G_2 to be the number of additional tosses required to obtain the next heads $\ldots G_r$ to be the number of additional tosses required to obtain the next [and rth] head. Then, $G_1, G_2, \ldots, G_n$ are independent random variables each with the Geomteric(p) distribution. Moreover,

$$X_r = G_1 + \cdots + G_r.$$

Therefore,

$$E(X_r) = E(G_1) + \cdots + E(G_r) = \frac{r}{p}, \quad \text{and}$$

$$\mathrm{Var}(X_r) = \mathrm{Var}(G_1) + \cdots + \mathrm{Var}(G_r) = \frac{rq}{p^2} \quad \Rightarrow \quad \mathrm{SD}(X_r) = \frac{\sqrt{rq}}{p}.$$

The Poisson distribution

We say that a random variable X has the *Poisson distribution with parameter* $\lambda > 0$ when

$$P\{X = k\} = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots.$$

Two questions arise: First, is the preceding a well-defined probability distribution? And second, why this particular form?

In order to answer the first question we have to prove that $\sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k! = 1$. But the Taylor's expansion of $f(x) = e^x$ tells us that

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

Divide by e^{λ} in order to find that $\sum_{k=0}^{\infty} P\{X = k\} = 1$, as one would like.

The next proposition [the law of rare events] shows how the Poisson distribution can be viewed as an approximation to the Binomial distribution when $p \propto 1/n$.

Proposition 1 (Poisson)**.** *Suppose λ is a positive constant that does not depend on n. If X has the Binomial distribution with parameters n and λ/n, then*

$$P\{X = k\} \approx \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for all } k = 0, \ldots, n,$$

provided that n is large.

Proof. The exact expression we want follows: For all $k = 0, \ldots, n$,

$$\begin{aligned} P\{X = k\} &= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}. \end{aligned}$$

Because k is held fixed, we have $n(n-1)\cdots(n-k+1) \approx n^k$ as $n \to \infty$. Therefore,

$$P\{X = k\} \approx \frac{1}{k!}\lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \approx \frac{1}{k!}\lambda^k \left(1 - \frac{\lambda}{n}\right)^{n},$$

since $(1 - \frac{\lambda}{n})^k \to 1$ as $n \to \infty$ [again because k is fixed]. It suffices to prove that

$$\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda} \qquad \text{as } n \to \infty.$$

Equivalently,

$$n \ln\left(1 - \frac{\lambda}{n}\right) \approx -\lambda \qquad \text{as } n \to \infty.$$

But this is clear because

$$\ln(1 - x) \approx 1 - x \qquad \text{if } x \approx 0,$$

thanks to the Taylor expansion of $\ln(1 - x)$. □

Now because Poisson(λ) is approximately Binomial $(n, \lambda/n)$, one might imagine that the expectation of a Poisson(λ) is approximately the expectation of Binomial $(n, \lambda/n)$ which is $n\lambda/n = \lambda$. And that the variance of a Poisson(λ) is approximately the variance of Binomial $(n\,, \lambda/n)$ which is $n(\lambda/n)(1-\frac{\lambda}{n}) \approx \lambda$. Although the preceding "argument" is logically flawed, it does produced the correct answers.

Let X have the Poisson(λ) distribution. In order to compute $E(X)$ we proceed as follows:

$$\begin{aligned} EX &= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!} \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^{j+1}}{j!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda. \end{aligned}$$

Similarly,

$$\begin{aligned} E(X^2) &= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} k(k-1) \frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=1}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} \\ &= \sum_{k=1}^{\infty} k(k-1) \frac{e^{-\lambda}\lambda^k}{k!} + \lambda \qquad \text{[from the computation for } EX]. \end{aligned}$$

Now

$$\begin{aligned} \sum_{k=1}^{\infty} k(k-1) \frac{e^{-\lambda}\lambda^k}{k!} &= e^{-\lambda}\lambda^2 \sum_{k=1}^{\infty} k(k-1) \frac{\lambda^{k-2}}{k!} \\ &= e^{-\lambda}\lambda^2 \frac{d^2}{d\lambda^2} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}\lambda^2 \frac{d^2}{d\lambda^2} e^{\lambda} = \lambda^2. \end{aligned}$$

Consequently,

$$E(X^2) = \lambda^2 + \lambda \qquad \Rightarrow \qquad \text{Var}(X) = \lambda \qquad \Leftrightarrow \qquad \text{SD}(X) = \sqrt{\lambda}.$$

Theorem 1 (A central limit theorem)**.** *Let X_λ have a Poisson distribution with parameter λ. Then the standardization of X_λ has approximately a standard normal distribution. That is, for all $-\infty \le a \le b \le \infty$,*

$$P\left\{ a \le \frac{X_\lambda - \lambda}{\sqrt{\lambda}} \le b \right\} \approx \Phi(b) - \Phi(a) \qquad \text{when } \lambda \text{ is large.}$$