

## 1. Combinatorics

Recall the two basic principles of counting [combinatorics]:

**First principle:**  $m$  distinct garden forks plus  $n$  distinct fish forks equals  $m + n$  distinct forks.

**Second principle:**  $m$  distinct knives and  $n$  distinct forks equals  $mn$  distinct ways of taking a knife and a fork.

## 2. Unordered Selection

**Example 6.1.** 6 dice are rolled. What is the probability that they all show different faces?

$$\Omega = ?$$

$$|\Omega| = 6^6.$$

If  $A$  is the event in question, then  $|A| = 6 \times 5 \times 4 \times 3 \times 2 \times 1$ .

**Definition 6.2.** If  $k$  is an integer  $\geq 1$ , then we define “ $k$  factorial” as the following integer:

$$k! = k \cdot (k - 1) \cdot (k - 2) \cdots 2 \cdot 1.$$

For consistency of future formulas, we define also

$$0! = 1.$$

**Example 6.3.** Five rolls of a fair die. What is  $P(A)$ , where  $A$  is the event that all five show different faces? Note that  $|A|$  is equal to 6 [which face is left out] times  $6^5$ . Thus,

$$P(A) = \frac{6 \cdot 5!}{6^5} = \frac{6!}{6^5}.$$

### 3. Ordered Selection

**Example 6.4.** Two-card poker.

$$P(\text{doubles}) = \frac{13 \times \binom{4 \times 3}{2}}{\binom{52 \times 51}{2}}.$$

**Theorem 6.5.**  $n$  objects are divided into  $r$  types.  $n_1$  are of type 1;  $n_2$  of type 2;  $\dots$ ;  $n_r$  are of type  $r$ . Thus,  $n = n_1 + \dots + n_r$ . Objects of the same type are indistinguishable. The number of permutations is

$$\binom{n}{n_1, \dots, n_r} = \frac{n!}{n_1! \dots n_r!}.$$

**Proof.** Let  $N$  denote the number of permutations; we seek to find  $N$ . For every permutation in  $N$  there are  $n_1! \dots n_r!$  permutations wherein all  $n$  objects are treated differently. Therefore,  $n_1! \dots n_r! N = n!$ . Solve to finish.  $\square$

**Example 6.6.**  $n$  people; choose  $r$  of them to form a “team.” The number of different teams is then

$$\frac{n!}{r!(n-r)!}.$$

You have to choose  $r$  of type 1 (“put this one in the team”), and  $n - r$  of type 2 (“leave this one out of the team”).

**Definition 6.7.** We write the preceding count statistic as “ $n$  choose  $r$ ,” and write it as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}.$$

**Example 6.8.** Roll 4 dice; let  $A$  denote the event that all faces are different. Then,

$$|A| = \binom{6}{4} 4! = \frac{6!}{2!} = \frac{6!}{2}.$$

The 6-choose-4 is there because that is how many ways we can choose the different faces. Thus,

$$P(A) = \frac{6!}{2 \times 4^6}.$$

**Example 6.9.** There are

$$\binom{52}{5} = 2,598,960$$

different standard poker hands possible.