Lecture 29

1. Marginals, distribution functions, etc.

If (X, Y) has joint density f, then

$$F_X(a) = \mathrm{P}\{X \le a\} = \mathrm{P}\{(X, Y) \in A\},$$

where $A = \{(x\, y) : x \le a\}$. Thus,

$$F_X(a) = \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f(x\,, y)\, dy \right) dx.$$

Differentiate, and apply the fundamental theorem of calculus, to find that

$$f_X(a) = \int_{-\infty}^{\infty} f(a\,, y)\, dy.$$

Similarly,

$$f_Y(b) = \int_{-\infty}^{\infty} f(x\,, b)\, dx.$$

Example 29.1. Let

$$f(x\,, y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$f_X(a) = \begin{cases} \int_0^a 8ay\, dy & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} 4a^3 & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

[Note the typo in the text, page 341.] Similarly,

$$f_Y(b) = \begin{cases} \int_b^1 8xb\,dx & \text{if } 0 < b < 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} 4b(1-b^2) & \text{if } 0 < b < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 29.2. Suppose (X, Y) is distributed uniformly on the square that joins the origin to the points $(1\,,0)$, $(1\,,1)$, and $(0\,,1)$. Then,

$$f(x\,,y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that X and Y are both distributed uniformly on $(0\,,1)$.

Example 29.3. Suppose (X, Y) is distributed uniformly in the circle of radius one about $(0\,,0)$. That is,

$$f(x\,,y) = \begin{cases} \dfrac{1}{\pi} & \text{if } x^2 + y^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$f_X(a) = \begin{cases} \displaystyle\int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \frac{1}{\pi} dy & \text{if } -1 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \dfrac{2}{\pi}\sqrt{1-a^2} & \text{if } -1 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

N.B.: f_Y is the same function. Therefore, in particular,

$$\begin{aligned} \mathrm{E}X &= \mathrm{E}Y \\ &= \frac{2}{\pi}\int_{-1}^{1} a\sqrt{1-a^2}\,da \\ &= 0, \qquad \text{by symmetry.} \end{aligned}$$

2. Functions of a random vector

Basic problem: If (X, Y) has joint density f, then what, if any, is the joint density of (U, V), where $U = u(X, Y)$ and $V = v(X, Y)$? Or equivalently, $(U, V) = T(X, Y)$, where

$$T(x\,, y) = \begin{pmatrix} u(x\,, y) \\ v(x\,, y) \end{pmatrix}.$$

Example 29.4. Let (X, Y) be distributed uniformly in the circle of radius $R > 0$ about the origin in the plane. Thus,

$$f_{X,Y}(x\,, y) = \begin{cases} \dfrac{1}{\pi R^2} & \text{if } x^2 + y^2 \leq R^2, \\ 0 & \text{otherwise.} \end{cases}$$

We wish to write (X, Y), in polar coordinates, as (R, Θ), where

$$R = \sqrt{X^2 + Y^2} \quad \text{and} \quad \Theta = \arctan(Y/X).$$

Then, we compute first the *joint distribution function* $F_{R,\Theta}$ of (R, Θ) as follows:

$$\begin{aligned} F_{R,\Theta}(a\,, b) &= \mathrm{P}\{R \leq a\,, \Theta \leq b\} \\ &= \mathrm{P}\{(X, Y) \in A\}, \end{aligned}$$

where A is the "partial cone" $\{(x\,, y) : x^2 + y^2 \leq a^2\,, \arctan(y/x) \leq b\}$. If a is not between 0 and R, or $b \notin (-\pi\,, \pi)$, then $F_{R,\Theta}(a\,, b) = 0$. Else,

$$\begin{aligned} F_{R,\Theta}(a\,, b) &= \iint_A f_{X,Y}(x\,, y)\, dx\, dy \\ &= \int_0^b \int_0^a \frac{1}{\pi R^2} r\, dr\, d\theta, \end{aligned}$$

after the change of variables $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. Therefore, for all $a \in (0\,, R)$ and $b \in (-\pi\,, \pi)$,

$$F_{R,\Theta}(a\,, b) = \begin{cases} \dfrac{a^2 b}{2\pi R^2} & \text{if } 0 < a < R \text{ and } -\pi < b < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$f_{R,\Theta}(a\,, b) = \frac{\partial^2 F_{R,\Theta}}{\partial a \partial b}(a\,, b).$$

Therefore,

$$f_{R,\Theta}(a\,, b) = \begin{cases} \dfrac{a}{\pi R^2} & \text{if } 0 < a < R \text{ and } -\pi < b < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The previous example can be generalized.

Suppose T is invertible with inverse function

$$T^{-1}(u\,,v) = \begin{pmatrix} x(u\,,v) \\ y(u\,,v) \end{pmatrix}.$$

The *Jacobian* of this transformation is

$$J(u\,,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 29.5. *If T is "nice," then*

$$f_{U,V}(u\,,v) = f_{X,Y}(x(u\,,v)\,,y(u\,,v))|J(u\,,v)|.$$

Example 29.6. In the polar coordinates example($r = u$, $\theta = v$),

$$\begin{aligned} r(x\,,y) &= \sqrt{x^2+y^2}, \\ \theta(x\,,y) &= \arctan(y/x) = \theta, \\ x(r\,,\theta) &= r\cos\theta, \\ y(r\,,\theta) &= r\sin\theta. \end{aligned}$$

Therefore, for all $r > 0$ and $\theta \in (-\pi\,,\pi)$,

$$\begin{aligned} J(r\,,\theta) &= (\cos(\theta) \times r\cos(\theta)) - (-r\sin(\theta) \times \sin(\theta)) \\ &= r\cos^2(\theta) + u\sin^2(\theta) = r. \end{aligned}$$

Hence,

$$f_{R,\Theta}(r\,,\theta) = \begin{cases} rf_{X,Y}(r\cos\theta\,,r\sin\theta) & \text{if } r > 0 \text{ and } \pi < \theta < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

You should check that this yields Example 29.4, for instance.

Example 29.7. Let us compute the joint density of $U = X$ and $V = X+Y$. Here,

$$\begin{aligned} u(x\,,y) &= x \\ v(x\,,y) &= x+y \\ x(u\,,v) &= u \\ y(u\,,v) &= v-u. \end{aligned}$$

Therefore,

$$J(u\,,v) = (1 \times 1) - (0 \times -1) = 1.$$

Consequently,

$$f_{U,V}(u\,,v) = f_{X,Y}(u\,,v-u).$$

This has an interesting by-product: The density function of $V = X + Y$ is

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(u\,,v)\,du \\ &= \int_{-\infty}^{\infty} f_{X,Y}(u\,,v-u)\,du. \end{aligned}$$