Lecture 26

1. Moment generating functions

Let X be a continuous random variable with density f. Its *moment generating function* is defined as

$$M(t) = \mathrm{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \tag{19}$$

provided that the integral exists.

Example 26.1 (Uniform$(0\,,1)$)**.** If $X = \text{Uniform}(0\,,1)$, then

$$M(t) = \mathrm{E}[e^{tX}] = \int_0^1 e^{tx}\,dx = \frac{e^t - 1}{t}.$$

Example 26.2 (Gamma)**.** If $X = \text{Gamma}(\alpha\,,\lambda)$, then

$$\begin{aligned} M(t) &= \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x}\,dx. \end{aligned}$$

If $t \geq \lambda$, then the integral is infinite. On the other hand, if $t < \lambda$, then

$$\begin{aligned} M(t) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{z^{\alpha-1}}{(\lambda-t)^{\alpha-1}} e^{-z} \frac{dz}{\lambda - t} \qquad (z = (\lambda-t)x) \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha) \times (\lambda-t)^\alpha} \underbrace{\int_0^\infty z^{\alpha-1} e^{-z}\,dz}_{\Gamma(\alpha)} \\ &= \frac{\lambda^\alpha}{(\lambda-t)^\alpha}. \end{aligned}$$

Thus,

$$M(t) = \begin{cases} \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha} & \text{if } t < \lambda, \\ \infty & \text{otherwise.} \end{cases}$$

Example 26.3 ($N(0\,,1)$)**.** If $X = N(0\,,1)$, then

$$\begin{aligned} M(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 - 2tx}{2}\right)\,dx \\ &= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 - 2tx + t^2}{2}\right)\,dx \\ &= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-t)^2}{2}\right)\,dx \\ &= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du \qquad (u = x - t) \\ &= e^{t^2/2}. \end{aligned}$$

2. Relation to moments

Suppose we know the function $M(t) = \mathrm{E}\exp(tX)$. Then, we can compute the moments of X from the function M by successive differentiation. For instance, suppose X is a continuous random variable with moment generating function M and density function f, and note that

$$M'(t) = \frac{d}{dt}\left(\mathrm{E}[e^{tX}]\right) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Now, if the integral converges absolutely, then a general fact states that we can take the derivative under the integral sign. That is,

$$M'(t) = \int_{-\infty}^{\infty} x e^{tx} f(x)\,dx = \mathrm{E}\left[Xe^{tX}\right].$$

The same end-result holds if X is discrete with mass function f, but this time,

$$M'(t) = \sum_x x e^{tx} f(x) = \mathrm{E}\left[Xe^{tX}\right].$$

Therefore, in any event:

$$M'(0) = \mathrm{E}[X].$$

In general, this procedure yields,

$$M^{(n)}(t) = \mathrm{E}\left[X^n e^{tX}\right].$$

Therefore,

$$M^{(n)}(0) = \mathrm{E}\left[X^n\right].$$

Example 26.4 (Uniform)**.** We saw earlier that if X is distributed uniformly on $(0\,,1)$, then for all real numbers t,

$$M(t) = \frac{e^t - 1}{t}.$$

Therefore,

$$M'(t) = \frac{te^t - e^t + 1}{t^2}, \quad M''(t) = \frac{t^2e^t - 2te^t + 2e^t - 2}{t^3},$$

whence

$$\mathrm{E}X = M'(0) = \lim_{t \searrow 0} \frac{te^t - e^t + 1}{t^2} = \lim_{t \searrow 0} \frac{te^t}{2t} = \frac{1}{2},$$

by l'Hopital's rule. Similarly,

$$\mathrm{E}\left[X^2\right] = \lim_{t \searrow 0} \frac{t^2e^t - 2te^t + 2e^t - 2}{t^3} = \lim_{t \searrow 0} \frac{t^2e^t}{3t^2} = \frac{1}{3}.$$

Alternatively, these can be checked by direct computation, using the fact that $\mathrm{E}[X^n] = \int_0^1 x^n\,dx = 1/(n+1)$.

A discussion of physical CLT machines . . . .