Lecture 19

1. Transformations of a mass function

1.1. The generating function. Recall that if X is an integer-valued random variable, then its [probability] *generating function*(p.g.f.) is

$$G(s) = E[s^X] = \sum_{k=-\infty}^{\infty} s^k f(k) \qquad \text{for all } -1 < s < 1.$$

1.2. The moment generating function. The *moment generating function* (m.g.f.) of a random variable X is

$$M(s) = E[e^{sX}] = \sum_x e^{sx} f(x),$$

provided that the sum exists.

This is indeed a useful transformation, viz.,

Theorem 19.1 (Uniqueness)**.** *If there exists* $s_0 > 0$ *such that* $M_X(s)$ *and* $M_Y(s)$ *are finite and equal for all* $s \in (-s_0, s_0)$*, then* $f_X = f_Y$*.*

Example 19.2. If

$$M(s) = \frac{1}{2}e^s + \frac{1}{4}e^{-\pi s} + \frac{1}{4}e^{es},$$

then M is an m.g.f. with

$$f(x) = \begin{cases} 1/2 & \text{if } x = 1, \\ 1/4 & \text{if } x = -\pi \text{ or } x = e, \\ 0 & \text{otherwise.} \end{cases}$$

2. Sums of independent random variables

Theorem 19.3. *If $X_1, \ldots, X_n$ are independent, with respective generating functions $G_{X_1}, \ldots, G_{X_n}$, then $\sum_{i=1}^n X_i$ has the p.g.f.,*

$$G(s) = G_{X_1}(s) \times \cdots \times G_{X_n}(s).$$

Proof. By induction, it suffices to do this for $n = 2$ (why?). But then

$$G_{X_1+X_2}(s) = E\left[s^{X_1+X_2}\right] = E\left[s^{X_1} \times s^{X_2}\right].$$

By independence, this is equal to the product of $E[s^{X_1}]$ and $E[s^{X_2}]$, which is the desired result. □

Example 19.4. Suppose $X = \text{bin}(n, p)$. Then we can write $X = I_1 + \cdots + I_n$, where $I_1, \ldots, I_n$ are independent, each taking the values zero (with probability $q = 1 - p$) and one (with probability p). Let us first compute

$$G_{I_j}(s) = E[s^{I_j}] = qs^0 + ps^1 = q + ps.$$

We can apply Theorem 19.3 then to find that

$$G_X(s) = (q + ps)^n.$$

Example 19.5. If $X = \text{bin}(n\,p)$ and $Y = \text{bin}(m, p)$ are independent, then by the previous example and Theorem 19.3,

$$G_{X+Y}(s) = (q + ps)^n (q + ps)^m = (q + ps)^{n+m}.$$

By the uniqueness theorem, $X + Y = \text{bin}(n + m, p)$. We found this out earlier by applying much harder methods. See Example 18.3.

Example 19.6. If $X = \text{Poisson}(\lambda)$, then

$$\begin{aligned} G(s) = E\left[s^X\right] &= \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}. \end{aligned}$$

The sum gives the Taylor expansion of $\exp(s\lambda)$. Therefore,

$$G(s) = \exp\left\{\lambda(s-1)\right\}.$$

Example 19.7. Now suppose $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\gamma)$ are independent. We apply the previous example and Theorem 19.3, in conjunction, to find that

$$\begin{aligned} G_{X+Y}(s) &= \exp\{\lambda(s-1)\}\exp\{\gamma(s-1)\} \\ &= \exp\{(\lambda+\gamma)(s-1)\}. \end{aligned}$$

Thus, $X + Y = \text{Poisson}(\gamma + \lambda)$, thanks to the uniqueness theorem and Example 19.6. For a harder derivation of the same fact see Example 17.4.

Next is another property of generating function, applied to random sums.

Theorem 19.8. *Suppose $X_0, X_1, X_2, \ldots$ and N are all independent, and $N \geqslant 0$. Suppose also that all X_is have the same distribution, with common p.g.f. G. Then, the p.g.f. of $S = \sum_{i=0}^{N} X_i$ is*

$$G_Z(s) = G_N(G(s)).$$

Proof. We know that

$$\begin{aligned} G_Z(s) = E\left[s^Z\right] &= \sum_{n=0}^{\infty} E\left(s^Z \mid N = n\right) P\{N = n\} \\ &= P\{N = 0\} + \sum_{n=1}^{\infty} E\left(s^{X_1+\cdots+X_n}\right) P\{N = n\}, \end{aligned}$$

by the independence of $X_1, X_2, \ldots$ and N. Therefore,

$$\begin{aligned} G_Z(s) &= \sum_{n=0}^{\infty} (G(s))^n P\{N = n\} \\ &= E\left[(G(s))^N\right], \end{aligned}$$

which is the desired result. □

3. Example: Branching processes

Branching processes are mathematical models for population genetics. The simplest branching process models asexual reproduction of genes, for example. It goes as follows: At time $n = 0$ there is one gene of a given (fixed) type. At time $n = 1$, this gene splits into a random number of "offspring genes." All subsequent genes split in the same way in time. We assume that all genes behave independently from all other genes, but the offspring distribution is the same for all genes as well. So here is the math model: Let $X_{i,j}$ be independent random variables, all with the same distribution (mass function). Let $Z_0 = 1$ be the population size at time 0,

and define $Z_1 = X_{1,1}$. This is the population size at time $n = 1$. Then, $Z_2 = \sum_{j=1}^{Z_1} X_{2,j}$ be the population size in generation 2, and more generally,

$$Z_n = \sum_{j=1}^{Z_{n-1}} X_{n,j}.$$

The big question of branching processes, and one of the big questions in population genetics, is "what happens to Z_n as $n \to \infty$"?

Let G denote the common generating function of the $X_{i,j}$'s, and let G_n denote the generating function of Z_n. Because $Z_0 = 1$, $G_0(s) = s$. Furthermore,

$$G_1(s) = E\left[s^{X_{1,1}}\right] = G(s) = G_0(G(s)).$$

In general,

$$G_{n+1}(s) = E\left[s^{\sum_{j=1}^{Z_n} X_{n+1,j}}\right] = G_n\left(G(s)\right),$$

thanks to Theorem 19.8. Because this is true for all $n \geqslant 0$, we have $G_1(s) = G(s)$,, $G_2(s) = G(G(s))$, and more generally,

$$G_k(s) = \overbrace{G(G(\cdots G(s)\cdots))}^{k \text{ times}} \qquad \text{for all } k \geqslant 0.$$

Note that $\{Z_n = 0\}$ is the event that the population has gone extinct by the nth generation. These events are increasing, therefore rule 4 of probabilities tells us that

$$P\{\text{ultimate extinction}\} = \lim_{n\to\infty} P\{Z_n = 0\}.$$

Theorem 19.9 (A. N. Kolmogorov)**.** *The extinction probability above is equal to the smallest nonnegative solution* s *to the equation*

$$G(s) = s.$$