By the Cauchy–Schwarz inequality, if $E(X^2) < \infty$, then EX is well defined and finite as well. In that case, the *variance* of X is defined as

$$\mathrm{Var}(X) = E(X^2) - |EX|^2 \,.$$

In order to understand why this means anything, note that

$$
\begin{aligned}
E\left[(X - EX)^2\right] = E\left[X^2 - 2XEX + (EX)^2\right] &= E(X^2) - 2E(X)E(X) + (EX)^2 \\
&= E(X^2) - |EX|^2 \\
&= \mathrm{Var}(X).
\end{aligned}
$$

Thus:

(1) We predict the as-yet-unseen value of X by the nonrandom number EX;

(2) $\mathrm{Var}(X)$ is the expected squared-error in this prediction. Note that $\mathrm{Var}(X)$ is also a nonrandom number.

1. Example 1

If $X = \mathrm{Bin}(n,p)$, then we have seen that $EX = np$ and $E(X^2) = (np)^2 + npq$. Therefore, $\mathrm{Var}(X) = npq$.

2. Example 2

Suppose X has mass function

$$
f(x) = \begin{cases} 1/4 & \text{if } x = 0, \\ 3/4 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}
$$

We saw in Lecture 11 that $EX = 3/4$. Now we compute the variance by first calculating

$$E(X^2) = \left(0^2 \times \frac{1}{4}\right) + \left(1^2 \times \frac{3}{4}\right) = \frac{3}{4}.$$

Thus,

$$\text{Var}(X) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{4}\left(1 - \frac{3}{4}\right) = \frac{3}{16}.$$

3. Example 3

Let n be a fixed positive integer, and X takes any of the values $1, \ldots, n$ with equal probability. Then, $f(x) = 1/n$ if $x = 1, \ldots, n$; $f(x) = 0$, otherwise. Let us calculate the first two "moments" of X.[1] In this way, we obtain the mean and the variance of X.

The first moment is the expectation, or the mean, and is

$$EX = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n} \times \frac{(n+1)n}{2} = \frac{n+1}{2}.$$

In order to compute $E(X^2)$ we need to know the algebraic identity:

$$\sum_{k=1}^{n} k^2 = \frac{(2n+1)(n+1)n}{6}. \tag{10}$$

This is proved by induction: For $n = 1$ it is elementary. Suppose it is true for $n - 1$. Then write

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(2(n-1)+1)(n-1+1)(n-1)}{6} + n^2,$$

thanks to the induction hypothesis. Simplify to obtain

$$\sum_{k=1}^{n} k^2 = \frac{(2n-1)n(n-1)}{6} + n^2 = \frac{(2n-1)(n^2-n)}{6} + n^2$$

$$= \frac{2n^3 - 3n^2 + n}{6} + \frac{6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6},$$

which easily yields (10).

Thus,

$$E(X^2) = \sum_{k=1}^{n} \frac{k^2}{n} = \frac{1}{n} \times \frac{(2n+1)(n+1)n}{6} = \frac{(2n+1)(n+1)}{6}.$$

---

[1]It may help to recall that the pth moment of X is $E(X^p)$.

Therefore,

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{(2n+1)(n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} \\
&= \frac{4n^2+6n+2}{12} - \frac{3n^2+6n+3}{12} \\
&= \frac{n^2-1}{12}.
\end{aligned}$$

4. Example 4

Suppose $X = \mathrm{Poisson}(\lambda)$. We saw in Lecture 10 that $EX = \lambda$. In order to compute $E(X^2)$, we first compute $E[X(X-1)]$ and find that

$$\begin{aligned}
E[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-2)!} \\
&= \lambda^2 \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^{k-2}}{(k-2)!}.
\end{aligned}$$

The sum is equal to one; change variables $(j = k-2)$ and recognize the jth term as the probability that $\mathrm{Poisson}(\lambda) = j$. Therefore,

$$E[X(X-1)] = \lambda^2.$$

Because $X(X-1) = X^2 - X$, the left-hand side is $E(X^2) - EX = E(X^2) - \lambda$. Therefore,

$$E(X^2) = \lambda^2 + \lambda.$$

It follows that

$$\mathrm{Var}(X) = \lambda.$$

5. Example 5

Suppose $f(x) = pq^{x-1}$ if $x = 1, 2, \ldots$; and $f(x) = 0$ otherwise. This is the Geometric(p) distribution. [The mass function for the first time to heads for a p-coin; see Lecture 8.] We have seen already that $EX = 1/p$ (Lecture 10). Let us find a new computation for this fact, and then go on and find also the variance.

$$\begin{aligned}
EX &= \sum_{k=1}^{\infty} kpq^{k-1} = p\sum_{k=1}^{\infty} kq^{k-1} \\
&= p\frac{d}{dq}\left(\sum_{k=0}^{\infty} q^k\right) = p\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p}.
\end{aligned}$$

Next we compute $E(X^2)$ by first finding

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)pq^{k-1} = \frac{p}{q} \sum_{k=1}^{\infty} k(k-1)q^{k-2}$$

$$= pq \frac{d^2}{dq^2} \left(\sum_{k=0}^{\infty} q^k \right) = \frac{p}{q} \frac{d^2}{dq^2} \left(\frac{1}{1-q} \right)$$

$$= pq \frac{d}{dq} \left(\frac{1}{(1-q)^2} \right) = pq \frac{2}{(1-q)^3} = \frac{2q}{p^2}.$$

Because $E[X(X-1)] = E(X^2) - EX = E(X^2) - (1/p)$, this proves that

$$E(X^2) = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{2-p}{p^2}.$$

Consequently,

$$\text{Var}(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}.$$

For a wholly different solution, see Example (13) on page 124 of your text.