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and is equal to 0 otherwise. The constant $B(a, b)$ is given by

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

The mean and variance of such a random variable are

$$E[X] = \frac{a}{a+b} \qquad \text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

PROBLEMS

1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

3. Consider the function

$$f(x) = \begin{cases} C(2x - x^3) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

Could f be a probability density function? If so, determine C. Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \dfrac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

(a) Find $P\{X > 20\}$.

(b) What is the cumulative distribution function of X?

(c) What is the probability that of 6 such types of devices at least 3 will function for at least 15 hours? What assumptions are you making?

5. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

what need the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

6. Compute $E[X]$ if X has a density function given by

(a) $f(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$;

(b) $f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$;

(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$.

7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \ge 0$$

Compute the expected lifetime of such a tube.

9. Consider Example 4b of Chapter 4, but now suppose that the seasonal demand is a continuous random variable having probability density function f. Show that the optimal amount to stock is the value s^* that satisfies

$$F(s^*) = \frac{b}{b + \ell}$$

where b is net profit per unit sale, ℓ is the net loss per unit unsold, and F is the cumulative distribution function of the seasonal demand.

10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

11. A point is chosen at random on a line segment of length L. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

12. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

13. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

14. Let X be a uniform (0, 1) random variable. Compute $E[X^n]$ by using Proposition 2.1 and then check the result by using the definition of expectation.

15. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$;
(b) $P\{4 < X < 16\}$;
(c) $P\{X < 8\}$;
(d) $P\{X < 20\}$;
(e) $P\{X > 16\}$.

16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

17. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

18. Suppose that X is a normal random variable with mean 5. If $P\{X > 9\} = .2$, approximately what is $\text{Var}(X)$?

19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P\{X > c\} = .10$.

20. If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

(a) at least 50 who are in favor of the proposition;
(b) between 60 and 70 inclusive who are in favor;
(c) fewer than 75 in favor.

21. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet 2 inches tall? What percentage of men in the 6-footer club are over 6 foot 5 inches?

22. The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = .9000$ and $\sigma = .0030$. The specification limits were given as $.9000 \pm .0050$.

(a) What percentage of forgings will be defective?
(b) What is the maximum allowable value of σ that will permit no more than 1 in 100 defectives When the widths are normally distributed with $\mu = .9000$ and σ?

23. One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that number 6 will appear between 150 and 200 times inclusively. If number 6 appears exactly 200 times, find the probability that number 5 will appear less than 150 times.

24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6?

25. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

26. Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

27. In 10,000 independent tosses of a coin, the coin landed heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.

28. Twelve percent of the population is lefthanded. Approximate the probability that there are at least 20 lefthanders in a school of 200 students. State your assumptions.

29. A model for the movement of a stock supposes that if the present price of the stock is s, then after one time period it will be either us with probability p, or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 time periods if $u = 1.012$, $d = 0.990$, and $p = .52$.

30. An image is partitioned into 2 regions—one white and the other black. A reading taken from a randomly chosen point in the white section will give a reading that is normally distributed with $\mu = 4$ and $\sigma^2 = 4$, whereas one taken from a randomly chosen point in the black region will have a normally distributed reading with parameters (6, 9). A point is randomly chosen on the image and has a reading of 5. If the fraction of the image that is black is α, for what value of α would the probability of making an error be the same whether one concluded the point was in the black region or in the white region?

31. **(a)** A fire station is to be located along a road of length A, $A < \infty$. If fires will occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to

$$\text{minimize } E[|X - a|]$$

when X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length—stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$, where X is now exponential with rate λ.

32. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is

(a) the probability that a repair time exceeds 2 hours;

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

33. The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

34. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over (0, 40).

35. The lung cancer hazard rate of a t-year-old male smoker, $\lambda(t)$, is such that

$$\lambda(t) = .027 + .00025(t - 40)^2 \qquad t \geq 40$$

Assuming that a 40-year-old male smoker survives all other hazards, what is the probability that he survives to (a) age 50 and (b) age 60 without contracting lung cancer?

36. Suppose that the life distribution of an item has hazard rate function $\lambda(t) = t^3$, $t > 0$. What is the probability that where p is their joint probability mass function. Also, if X and Y are jointly continuous with joint density function f, then the *conditional probability density function* of X given that $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

The ordered values $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ of a set of independent and identically distributed random variables are called the *order statistics* of that set. If the random variables are continuous with density function f, then the joint density function of the order statistics is

$$f(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) \qquad x_1 \le x_2 \le \cdots \le x_n$$

The random variables $X_1, \ldots, X_n$ are called *exchangeable* if the joint distribution of $X_{i_1}, \ldots, X_{i_n}$ is the same for every permutation $i_1, \ldots, i_n$ of $1, \ldots, n$.

PROBLEMS

1. Two fair dice are rolled. Find the joint probability mass function of X and Y when
(a) X is the largest value obtained on any die and Y is the sum of the values;
(b) X is the value on the first die and Y is the larger of the two values;
(c) X is the smallest and Y is the largest value obtained on the dice.

2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
(a) X_1, X_2;
(b) X_1, X_2, X_3.

3. In Problem 2, suppose that the white balls are numbered, and let Y_i equal 1 if the ith white ball is selected and 0 otherwise. Find the joint probability mass function of
(a) Y_1, Y_2;
(b) Y_1, Y_2, Y_3.

4. Repeat Problem 2 when the ball selected is replaced in the urn before the next selection.

5. Repeat Problem 3a when the ball selected is replaced in the urn before the next selection.

6. A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by N_1 the number of tests made until the first defective is identified and by N_2 the number of additional tests until the second defective is identified; find the joint probability mass function of N_1 and N_2.

7. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

8. The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y,\ 0 < y < \infty$$

(a) Find c.
(b) Find the marginal densities of X and Y.
(c) Find $E[X]$.

9. The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$

(a) Verify that this is indeed a joint density function.
(b) Compute the density function of X.
(c) Find $P\{X > Y\}$.
(d) Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.
(e) Find $E[X]$.
(f) Find $E[Y]$.

10. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 \le x < \infty, 0 \le y < \infty$$

Find (a) $P\{X < Y\}$ and (b) $P\{X < a\}$.

11. A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 plasma set on that day?

12. The number of people that enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

13. A man and a woman agree to meet at a certain location about 12:30 P.M. If the man arrives at a time uniformly distributed between 12:15 and 12:45 and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 P.M., find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

14. An ambulance travels back and forth, at a constant speed, along a road of length L. At a certain moment of time an accident occurs at a point uniformly distributed on the road. [That is, its distance from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, compute, assuming independence, the distribution of its distance from the accident.

15. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

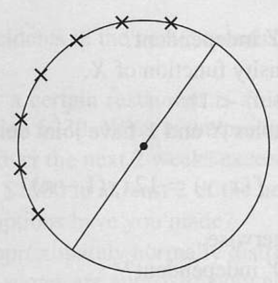
$$f(x, y) = \begin{cases} c & \text{if } (x, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $1/c =$ area of region R.

Suppose that (X, Y) is uniformly distributed over the square centered at $(0, 0)$, whose sides are of length 2.

(b) Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.
(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \le 1\}$.

16. Suppose that n points are independently chosen at random on the perimeter of a circle, and we want the probability that they all lie in some semicircle. (That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line.)



Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for $180°$, $i = 1, \ldots, n$.

(a) Express A in terms of the A_i.
(b) Are the A_i mutually exclusive?
(c) Find $P(A)$.

17. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3?

18. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

19. Show that $f(x, y) = 1/x$, $0 < y < x < 1$ is a joint density function. Assuming that f is the joint density function of X, Y, find

(a) the marginal density of Y;
(b) the marginal density of X;
(c) $E[X]$;
(d) $E[Y]$.

20. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? What if $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

21. Let

$$f(x, y) = 24xy \qquad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1$$

and let it equal 0 otherwise.

(a) Show that $f(x, y)$ is a joint probability density function.
(b) Find $E[X]$.
(c) Find $E[Y]$.

22. The joint density function of X and Y is

$$f(x, y) = \begin{cases} x + y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?
(b) Find the density function of X.
(c) Find $P\{X + Y < 1\}$.

23. The random variables X and Y have joint density function.

$$f(x, y) = 12xy(1 - x) \qquad 0 < x < 1,\ 0 < y < 1$$

and equal to 0 otherwise.
(a) Are X and Y independent?
(b) Find $E[X]$.
(c) Find $E[Y]$.
(d) Find $\text{Var}(X)$.
(e) Find $\text{Var}(Y)$.

24. Consider independent trials each of which results in outcome $i, i = 0, 1, \ldots, k$ with probability p_i, $\sum_{i=0}^{k} p_i = 1$. Let N denote the number of trials needed to obtain an outcome that is not equal to 0, and let X be that outcome.
(a) Find $P\{N = n\}, n \geq 1$.
(b) Find $P\{X = j\}, j = 1, \ldots, k$.
(c) Show that $P\{N = n, X = j\} = P\{N = n\}P\{X = j\}$.
(d) Is it intuitive to you that N is independent of X?
(e) Is it intuitive to you that X is independent of N?

25. Suppose that 10^6 people arrive at a service station at times that are independent random variables, each of which is uniformly distributed over $(0, 10^6)$. Let N denote the number that arrive in the first hour. Find an approximation for $P\{N = i\}$.

26. Suppose that A, B, C, are independent random variables, each being uniformly distributed over (0, 1).
(a) What is the joint cumulative distribution function of A, B, C?
(b) What is the probability that all of the roots of the equation $Ax^2 + Bx + C = 0$ are real?

27. If X is uniformly distributed over (0, 1) and Y is exponentially distributed with parameter $\lambda = 1$, find the distribution of (a) $Z = X + Y$ and (b) $Z = X/Y$. Assume independence.

28. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $P\{X_1 < X_2\}$.

29. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by $W = I^2R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1 - x) \qquad 0 \leq x \leq 1$$
$$f_R(x) = 2x \qquad 0 \leq x \leq 1$$

Determine the density function of W.

30. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that an article of 10 pages contains (a) 0, and (b) 2 or more typographical errors? Explain your reasoning!

31. The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be
(a) more than 2 such accidents in the next month;
(b) more than 4 such accidents in the next 2 months;

(c) more than 5 such accidents in the next 3 months?
Explain your reasoning!

32. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that
  (a) the total gross sales over the next 2 weeks exceeds \$5000;
  (b) weekly sales exceed \$2000 in at least 2 of the next 3 weeks?
What independence assumptions have you made?

33. Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, then assuming that their scores are independent random variables, approximate the probability that
  (a) Jack's score is higher;
  (b) the total of their scores is above 350.

34. According to the U.S. National Center for Health Statistics, 25.2 percent of males and 23.6 percent of females never eat breakfast. Suppose that random samples of 200 men and 200 women are chosen. Approximate the probability that
  (a) at least 110 of these 400 people never eat breakfast;
  (b) the number of the women who never eat breakfast is at least as large as the number of the men who never eat breakfast.

35. In Problem 2, calculate the conditional probability mass function of X_1 given that
  (a) $X_2 = 1$;
  (b) $X_2 = 0$.

36. In Problem 4, calculate the conditional probability mass function of X_1 given that
  (a) $X_2 = 1$;
  (b) $X_2 = 0$.

37. In Problem 3, calculate the conditional probability mass function of Y_1 given that
  (a) $Y_2 = 1$;
  (b) $Y_2 = 0$.

38. In Problem 5, calculate the conditional probability mass function of Y_1 given that
  (a) $Y_2 = 1$;
  (b) $Y_2 = 0$.

39. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, \ldots, X\}$. Call this second number Y.
  (a) Find the joint mass function of X and Y.
  (b) Find the conditional mass function of X given that $Y = i$. Do it for $i = 1, 2, 3, 4, 5$.
  (c) Are X and Y independent? Why?

40. Two dice are rolled. Let X and Y denote, respectively, the largest and smallest values obtained. Compute the conditional mass function of Y given $X = i$, for $i = 1, 2, \ldots, 6$. Are X and Y independent? Why?

41. The joint probability mass function of X and Y is given by

$$p(1,1) = \tfrac{1}{8} \qquad p(1,2) = \tfrac{1}{4}$$
$$p(2,1) = \tfrac{1}{8} \qquad p(2,2) = \tfrac{1}{2}$$

  (a) Compute the conditional mass function of X given $Y = i$, $i = 1, 2$.
  (b) Are X and Y independent?
  (c) Compute $P\{XY \le 3\}$, $P\{X + Y > 2\}$, $P\{X/Y > 1\}$.

42. The joint density function of X and Y is given by

$$f(x, y) = xe^{-x(y+1)} \quad x > 0, \ y > 0$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.
(b) Find the density function of $Z = XY$.

43. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x} \quad 0 \le x < \infty, \ -x \le y \le x$$

Find the conditional distribution of Y, given $X = x$.

44. An insurance company supposes that each person has an accident parameter and that the yearly number of accidents of someone whose accident parameter is λ is Poisson distributed with mean λ. They also suppose that the parameter value of a newly insured person can be assumed to be the value of a gamma random variable with parameters s and α. If a newly insured person has n accidents in her first year, find the conditional density of her accident parameter. Also, determine the expected number of accidents that she will have in the following year.

45. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over (0, 1), compute the probability that the largest of the three is greater than the sum of the other two.

46. A complex machine is able to operate effectively as long as at least 3 of its 5 motors are functioning. If each motor independently functions for a random amount of time with density function $f(x) = xe^{-x}, x > 0$, compute the density function of the length of time that the machine functions.

47. If 3 trucks break down at points randomly distributed on a road of length L, find the probability that no 2 of the trucks are within a distance d of each other when $d \le L/2$.

48. Consider a sample of size 5 from a uniform distribution over (0, 1). Compute the probability that the median is in the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$.

49. If X_1, X_2, X_3, X_4, X_5 are independent and identically distributed exponential random variables with the parameter λ, compute
(a) $P\{\min(X_1, \ldots, X_5) \le a\}$;
(b) $P\{\max(X_1, \ldots, X_5) \le a\}$.

50. Derive the distribution of the range of a sample of size 2 from a distribution having density function $f(x) = 2x, 0 < x < 1$.

51. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x, y) = \frac{1}{\pi} \quad x^2 + y^2 \le 1$$

Find the joint density function of the polar coordinates $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1} Y/X$.

52. If X and Y are independent random variables both uniformly distributed over (0, 1), find the joint density function of $R = \sqrt{X^2 + Y^2}$, $\Theta = \tan^{-1} Y/X$.

53. If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with rate 1, show directly (without using the results of Example 7b) that X and Y defined by

$$X = \sqrt{2Z} \cos U$$
$$Y = \sqrt{2Z} \sin U$$

are independent standard normal random variables.

54. X and Y have joint density function

$$f(x, y) = \frac{1}{x^2 y^2} \qquad x \geq 1,\ y \geq 1$$

(a) Compute the joint density function of $U = XY, V = X/Y$.
(b) What are the marginal densities?

55. If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of
(a) $U = X + Y, V = X/Y$;
(b) $U = X, V = X/Y$;
(c) $U = X + Y, V = X/(X + Y)$.

56. Repeat Problem 55 when X and Y are independent exponential random variables, each with parameter $\lambda = 1$.

57. If X_1 and X_2 are independent exponential random variables each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

58. If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}, 0 < x < \infty$, derive the joint distribution of $U = X + Y,\ V = X + Z,\ W = Y + Z$.

59. In Example 8b, let $Y_{k+1} = n + 1 - \sum_{i=1}^{k} Y_i$. Show that $Y_1, \ldots, Y_k, Y_{k+1}$ are exchangeable. Note that Y_{k+1} is the number of balls one must observe to obtain a special ball if one considers the balls in their reverse order of withdrawal.

60. Consider an urn containing n balls, numbered $1, \ldots, n$, and suppose that k of them are randomly withdrawn. Let X_i equal 1 if ball numbered i is removed and let it be 0 otherwise. Show that $X_1, \ldots, X_n$ are exchangeable.

THEORETICAL EXERCISES

1. Verify Equation (1.2).

2. Suppose that the number of events that occur in a given time period is a Poisson random variable with parameter λ. If each event is classified as a type i event with probability $p_i, i = 1, \ldots, n, \sum p_i = 1$, independently of other events, show that the numbers of type i events that occur, $i = 1, \ldots, n$, are independent Poisson random variables with respective parameters $\lambda p_i, i = 1, \ldots, n$.

3. Suggest a procedure for using Buffon's needle problem to estimate π. Surprisingly enough, this was once a common method of evaluating π.

4. Solve Buffon's needle problem when $L > D$.
ANSWER: $\frac{2L}{\pi D}(1 - \sin\theta) + 2\theta/\pi$, where θ is such that $\cos\theta = D/L$.

5. If X and Y are independent continuous positive random variables, express the density function of (a) $Z = X/Y$ and (b) $Z = XY$ in terms of the density functions of X and Y. Evaluate these expressions in the special case where X and Y are both exponential random variables.

6. Show analytically (by induction) that $X_1 + \cdots + X_n$ has a negative binomial distribution when the $X_i, i = 1, \ldots, n$ are independent and identically distributed geometric random variables. Also, give a second argument that verifies the preceding without any need for computations.

7. (a) If X has a gamma distribution with parameters (t, λ), what is the distribution of $cX, c > 0$?