Solutions to Midterm 2
Mathematics 5010–001, Summer 2009

1. *There are 2 red bottles and 3 green bottles in a basket. We select a bottle at random, independently, until we sample a red bottle. Compute the mass function of* X*, where* X *denote the sample size required to select the first red bottle.*

   Solution: X has a geometric distribution with parameter $p = 2/5$. Therefore,

$$p(x) = \begin{cases} \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{x-1} & \text{if } x = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

2. *The following game is called "wheel of fortune," and is popular in many carnivals and casinos: A player bets on an integer between 1 and 6. Three dice are then rolled. If the number bet by the player appears* i *times* $[i = 1, 2, 3]$*, then the player wins* i *units. Else, the player loses one unit. Let* X *denote the resulting fortune of the player. Assuming that the dice are rolled independently, compute* EX.

   Solution: The possible values are manifestly -1, 1, 2, and 3. Note that $P\{X = j\}$ is the probability that a binomial [parameters $n = 3$ and $p = 1/6$] is equal to j, when $j = 1, 2, 3$. And $P\{X = -1\}$ is the probability that the same binomial is zero. That is,

$$\begin{aligned} f(-1) &= \binom{3}{0}\left(\frac{1}{6}\right)^0\left(1-\frac{1}{6}\right)^{3-0} = \frac{125}{216} \\ f(1) &= \binom{3}{1}\left(\frac{1}{6}\right)^1\left(1-\frac{1}{6}\right)^{3-1} = \frac{75}{216} \\ f(2) &= \binom{3}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{3-2} = \frac{15}{216} \\ f(3) &= \binom{3}{3}\left(\frac{1}{6}\right)^3\left(1-\frac{1}{6}\right)^{3-3} = \frac{1}{216}. \end{aligned}$$

   Consequently,

$$EX = \left(-1 \times \frac{125}{216}\right) + \left(1 \times \frac{75}{216}\right) + \left(2 \times \frac{15}{216}\right) + \left(3 \times \frac{1}{216}\right) = -\frac{17}{216}.$$

3. *Suppose* X *has the Poisson distribution with parameter* 2*. That is, suppose the*

mass function of X *is*

$$f(k) = \begin{cases} \dfrac{e^{-2}2^k}{k!} & \textit{if } k = 0, 1, 2, \ldots, \\ \\ 0 & \textit{otherwise}. \end{cases}$$

What is the maximum value of f*, and for which values* x *is* $f(x)$ *equal to* $\max f$*?*

Solution: We follow the hint and compute

$$\frac{f(k+1)}{f(k)} = \frac{2}{k+1} \qquad \text{for all } k \geqslant 0.$$

which is $\leqslant 1$ for all values of $k \geqslant 1$. It follows that $f(1)/f(0) = 2$ and $f(k+1) \leqslant f(k)$ for $k \geqslant 1$. Thus, $f(x)$ is maximized at $x = 1$ with $f(1) = 2e^{-2}$. It is also maximized at $x = 2$; in either case, $\max_x f(x) = 2e^{-2}$.

4. *I have two coins: One is two-headed; the other is fair. I select one coin at random (both coins equally likely), and toss it 100 times independently. Suppose I tell you that all tosses resulted in heads. Given this information, what would you say the odds are that I had selected the fair coin?*

Solution: Let F denote the event that I choose the fair coin. Let H denote the event that I toss 100 heads. We want $P(F \mid H)$. Here is what we know:

$$P(H \mid F) = \left(\frac{1}{2}\right)^{100}, \quad P(H \mid F^c) = 1, \quad P(F) = \frac{1}{2}.$$

Therefore, we may write [using Bayes's rule]:

$$P(F \mid H) = \frac{P(H \mid F)P(F)}{P(H \mid F)P(F) + P(H \mid F^c)P(F^c)} = \frac{(\frac{1}{2})^{100}(\frac{1}{2})}{(\frac{1}{2})^{100}(\frac{1}{2}) + (\frac{1}{2})} = \frac{1}{2}\left(\frac{(1/2)^{100}}{(1/2)^{100} + 1}\right).$$

Because $(1/2)^{100} + 1 \approx 1$, this probability is very close to 2^{-101}.