Partial Solutions to Homework 5
Mathematics 5010–1, Summer 2009

Problem 9, p. 303. Here is the table: the $(a\,,b)$ entry is $f(a\,,b)$.

y/x	a	b	c
a	0	1/6	1/6
b	1/6	0	1/6
c	1/6	1/6	0

Problem 15, p. 303. A detailed solution can be found in the back of your text.

Problem 16, p. 303. The formula for $f_X(x)$ follows from $P\{UV \leqslant x\}$ and the fundamental theorem of calculus. In order to find $P\{UV \leqslant x\}$, note that

$$P\{UV \leqslant x\} = P\{UV \leqslant x\,, V > 0\} + P\{UV \leqslant x\,, V < 0\},$$

and then compute each integral as a double sum.

Problem 21, p. 304. First U:

$$P\{U > k\} = P\{X > k\,, Y > k\} = P\{X > k\}P\{Y > k\}.$$

Now, for $k = 1\,, \ldots\,, n-1$,

$$P\{X > k\} = P\{Y > k\} = 1 - P\{Y \leqslant k\} = 1 - \frac{k}{n}.$$

Therefore,

$$P\{U > k\} = \left(1 - \frac{k}{n}\right)^2 \qquad \text{for } k = 1, \ldots, n-1.$$

And

$$P\{U = m\} = P\{U > m-1\} - P\{U > m\} = \left(1 - \frac{m-1}{n}\right)^2 - \left(1 - \frac{m}{n}\right)^2$$
$$= \frac{2n + 2m - 1}{n^2} \qquad \text{for } m = 1, \ldots, n-1.$$

And $P\{U = n\} = P\{X = n\}P\{Y = n\} = 1/n^2$. Therefore,

$$EU = \sum_{m=1}^{n-1} m\left(\frac{2n+2m-1}{n^2}\right) + \frac{1}{n}$$
$$= \frac{1}{n^2}\left((2n-1)\sum_{m=1}^{n-1} m + 2\sum_{m=1}^{n-1} m^2\right) + \frac{1}{n},$$

and then plug the known formulas for $\sum_{m=1}^{n-1} m$ and $\sum_{m=1}^{n-1} m^2$ into this.

To compute $E(U^2)$—and hence $\mathrm{var}(U)$—we do the same thing as above, but start with

$$E(U^2) = \sum_{m=1}^{n-1} m^2 \left(\frac{2n+2m-1}{n^2}\right) + 1.$$

The other computations are similar, but we need $P\{V = k\}$. That is computed slightly differently: For all $k = 1, \ldots, n$,

$$P\{V \leqslant k\} = P\{X \leqslant k\}P\{Y \leqslant k\} = \left(\frac{k}{n}\right)^2.$$

Therefore,

$$P\{V = k\} = P\{V \leqslant k\} - P\{V \leqslant k-1\} = \left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2 = \frac{2k-1}{n^2}.$$

Now we can proceed as before.

Problem 24, p. 304. First we need F_Y. If $a \leqslant 0$ then $F_Y(a) = 0$. If $a > 0$, then

$$F_Y(a) = P\left\{e^X \leqslant a\right\} = P\{X \leqslant \ln a\} = F_X(\ln a).$$

Therefore, $f_Y(a) = 0$ if $a \leqslant 0$ and

$$f_Y(a) = \frac{f_X(\ln a)}{a} = \frac{e^{-(\ln a)^2/2}}{a\sqrt{2\pi}} \qquad \text{if } a > 0.$$

In particular,

$$\begin{aligned} E(Y) &= \int_0^\infty \frac{e^{-(\ln a)^2/2}}{\sqrt{2\pi}}\, da = \int_{-\infty}^\infty \frac{e^{-z^2/2}\, e^z}{\sqrt{2\pi}}\, dz \\ &= \int_{-\infty}^\infty \frac{e^{-\frac{1}{2}(z^2-2z)}}{\sqrt{2\pi}}\, dz = e^{1/2} \int_{-\infty}^\infty \frac{e^{-\frac{1}{2}(z^2-2z+1)}}{\sqrt{2\pi}}\, dz. \end{aligned}$$

We have been completing the square. Now change variables one more time:

$$E(Y) = e^{1/2} \int_{-\infty}^\infty \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}}\, dw = e^{1/2}.$$

Similarly,

$$\begin{aligned} E(Y^2) &= \int_0^\infty \frac{a e^{-(\ln a)^2/2}}{\sqrt{2\pi}}\, da = \int_0^\infty \frac{e^{2z} e^{-z^2/2}}{\sqrt{2\pi}}\, dz \\ &= \int_0^\infty \frac{e^{-\frac{1}{2}(z^2-4z)}}{\sqrt{2\pi}}\, dz = e^2 \int_0^\infty \frac{e^{-\frac{1}{2}(z-2)^2}}{\sqrt{2\pi}}\, dz = e^2. \end{aligned}$$

Therefore, $\mathrm{var}(Y) = e^2 - e = e(e-1)$.

Problem 25, p. 305. First, we note that [logically speaking], $N \geqslant n$ means that $X_0 = \max(X_1, \ldots, X_{n-1})$. Therefore,

$$P\{N \geqslant n\} = P\{X_0 \geqslant X_1, \cdots, X_0 \geqslant X_{n-1}\} \geqslant P\{X_0 > X_1, \cdots, X_0 > X_{n-1}\}.$$

Let A_j denote the event that X_j is the unique maximum among $X_0, \ldots, X_{n-1}$. Note that: (i) A_j's are disjoint; and (ii) $P(A_0) = P(A_1) = \ldots = P(A_{n-1})$. Because $P(A_0) + \cdots + P(A_{n-1}) = P(A_1 \cup \cdots \cup A_{n-1}) = 1$, this shows that $P(A_0) = 1/n$. And therefore, $P\{N \geqslant n\} \geqslant n^{-1}$.