Partial Solutions to Homework 4
Mathematics 5010–1, Summer 2009

Problem 3, p. 234. $f(x) = cx^{-d}$ if $x \geqslant 1$ and $f(x) = 0$ if $x < 1$.

(a)

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_1^{\infty} x^{-d}\,dx = \begin{cases} \frac{c}{d-1} & \text{if } d > 1, \\ \infty & \text{if } d \leqslant 1. \end{cases}$$

So f is a probability density if and only if $d > 1$. And when $d > 1$, we have $c = d - 1$.

(b) $EX = (d-1)\int_1^\infty x^{-d+1}\,dx$. This is $= +\infty$ if $d \leqslant 2$. But if $d > 2$, then $EX = (d-1)/(d-2)$.

(c) $E(X^2) = (d-1)\int_1^\infty x^{-d+2}\,dx$. This is $= +\infty$ if $d \leqslant 3$. But if $d > 3$, then $E(X^2) = (d-1)/(d-3)$. Therefore,

$$\text{Var}(X) = \frac{d-1}{d-3} - \left(\frac{d-1}{d-2}\right)^2 \quad \text{if } d > 3.$$

If $2 < d \leqslant 3$, then the variance is infinite; else it is not defined.

Problem 4, p. 234. In this problem, $f(x) = \lambda e^{-\lambda x}$ if $x \geqslant 0$, and $f(x) = 0$ if $x < 0$. The constant λ is a fixed positive parameter.

(a) You need to know that $\arcsin(1/2) = \pi/6 \approx 0.524$. Now "$\sin X > \frac{1}{2}$" means that:

i. Either X is between $\frac{\pi}{6}$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$;
ii. Or X is between $2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$ and $3\pi - \frac{\pi}{6} = \frac{17\pi}{6}$;
iii. Or ....

[The general term is that X must be between $2n\pi + \frac{\pi}{6}$ and $(2n+1)\pi - \frac{\pi}{6}$ for some integer $n \geqslant 0$.] These events are all disjoint. Therefore,

$$\begin{aligned} P\left\{\sin X > \tfrac{1}{2}\right\} &= \sum_{n=0}^{\infty} \int_{2n\pi+(\pi/6)}^{(2n+1)\pi-(\pi/6)} \lambda e^{-\lambda x}\,dx \\ &= \sum_{n=0}^{\infty} \left(e^{-\lambda\{(2n\pi+(\pi/6))\}} - e^{-\lambda\{(2n+1)\pi-(\pi/6))\}} \right) \\ &= \sum_{n=0}^{\infty} e^{-2n\pi\lambda} \left(e^{-\lambda\pi/6} - e^{-5\lambda\pi/6} \right) \\ &= \frac{e^{-\lambda\pi/6} - e^{-5\lambda\pi/6}}{1 - e^{-2\pi\lambda}}; \end{aligned}$$

the last lines holds because of the formula for geometric series.

(b) $E(X^n) = \int_0^\infty x^n \lambda e^{-\lambda x}\, dx = \lambda^{-n} \int_0^\infty z^n e^{-z}\, dz$ after a change of variables. Note that $\int_0^\infty z^n e^{-z}\, dz = \Gamma(n+1) = n!$. Therefore, $E(X^n) = n!/\lambda^n$ for all $n \geqslant 1$.

Problem 6, p. 234. (a) Want $\int_0^1 cx(1-x)\, dx = 1$, but $\int_0^1 x(1-x)\, dx = \frac{1}{6}$. Therefore, $c = 6$. And $F(a) = 1$ if $a > 1$ and $F(a) = 0$ if $a < 0$. So it remains to compute $F(a)$ for $0 \leqslant a \leqslant 1$. But in that case,

$$F(a) = \int_0^a 6x(1-x)\, dx = 3a^2 - 2a^3 = a^2(3-2a).$$

(b) This is not a density function because $\int_1^\infty x^{-1}\, dx = \infty$.

(c) Note that $-x^2 + 4x = 4 - (x-2)^2$ [complete the square]. Therefore,

$$\int_{-\infty}^\infty e^{-x^2+4x}\, dx = e^4 \int_{-\infty}^\infty e^{-(x-2)^2}\, dx = e^4 \int_{-\infty}^\infty e^{-y^2}\, dy = \frac{e^4}{\sqrt{2}} \int_{-\infty}^\infty e^{-z^2/2}\, dz,$$

which is $e^4\sqrt{\pi}$. Therefore, $c = e^{-4}\pi^{-1/2}$. And

$$F(a) = \frac{1}{e^4\sqrt{\pi}} \int_{-\infty}^a e^{-x^2+4x}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^a e^{-(x-2)^2}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{a-2} e^{-y^2}\, dy.$$

Another change of variables yields: For all $-\infty < a < \infty$,

$$F(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}(a-2)} e^{-z^2/2}\, dz = \Phi\left(\sqrt{2}(a-2)\right).$$

(d) Note that

$$\int_{-\infty}^\infty \frac{e^x}{(1+e^x)^2}\, dx = \int_0^\infty \frac{1}{(1+z)^2}\, dz = 1.$$

Therefore, $c = 1$. And for all $-\infty < a < \infty$,

$$F(a) = \int_{-\infty}^a \frac{e^x}{(1+e^x)^2}\, dx = \int_0^{e^a} \frac{1}{(1+z)^2}\, dz = \frac{e^a}{1+e^a}.$$

Problem 10, p. 234. To be concrete, let us suppose that the dart board is the disc of radius $r > 0$ around $(0,0)$. Uniform distribution implies that $f(x,y) = 1/(\pi r^2)$ if $x^2 + y^2 \leqslant r^2$; and $f(x,y) = 0$ otherwise.

If $-r \leqslant a \leqslant r$, then "$Y > a$" means that the point $(X,Y) \in A$, where A denotes the portion of the circle that lies above the line $y = a$. Therefore, in that case,

$$P\{Y > a\} = \int_a^r \left(\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx \right) dy = \frac{2}{\pi r^2} \int_a^r \sqrt{r^2 - y^2}\, dy.$$

Change variables $[y = rz]$:

$$P\{Y > a\} = \frac{2}{\pi r}\int_{a/r}^{1} \sqrt{r^2 - r^2 z^2}\, dz = \frac{2}{\pi}\int_{a/r}^{1} \sqrt{1 - z^2}\, dz.$$

Therefore we have a standard calculus exercise: Change variables $[z := \sin\theta]$ to find that

$$P\{Y > a\} = \frac{2}{\pi}\int_{\arcsin(a/r)}^{\pi/2} (\cos\theta)^2\, d\theta.$$

Use the half-angle formula: $(\cos\theta)^2 = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ to find that

$$\begin{aligned} P\{Y > a\} &= \frac{1}{\pi}\int_{\arcsin(a/r)}^{\pi/2} \left(1 + \cos(2\theta)\right)\, d\theta \\ &= \frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{a}{r}\right) - \frac{1}{2\pi}\sin\left(2\arcsin\left(\frac{a}{r}\right)\right). \end{aligned}$$

Consequently, if $|a| \leqslant r$ then

$$F_Y(a) = 1 - P\{Y > a\} = \frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{a}{r}\right) + \frac{1}{2\pi}\sin\left(2\arcsin\left(\frac{a}{r}\right)\right).$$

If $a > r$, then $F_Y(a) = 1$; and if $a < -r$, then $F_Y(a) = 0$.

To compute $f_Y(a)$ we differentiate F_Y: If $|a| > r$ then $f_Y(a) = 0$. Otherwise, if $|a| \leqslant r$ then

$$f_Y(a) = \frac{1}{\pi}\frac{1/r}{1 - (a/r)^2} + \frac{1}{2\pi}\cos\left(2\arcsin\left(\frac{a}{r}\right)\right)\cdot\frac{2/r}{1 - (a/r)^2}.$$

The second part of the problem [that is, the computation of F_R, f_R, and ER] was done in the lectures.

Problem 26, p. 236. I will do this in the discrete case. The continuous case is proved similarly. Let $\mu := EX$ and recall that

$$\text{var}X = E\left(|X - \mu|^2\right) = \sum_k |k - \mu|^2 f(k).$$

This is assumed to be zero. But it is a sum of nonnegative numbers. Therefore, $f(k) = 0$ whenever $k \neq \mu$. This implies that $f(\mu) = 1$, and hence $P\{X = \mu\} = 1$. In other words, the problem is valid for $a = EX$.

Problem 3, p. 302. It is easy to see that $\min\{X, Y\} = 1$; this is because one of the two random variables is always one [after all, the first toss is either the first head, or the first tail!]. Therefore, $E(\min\{X, Y\}) = 1$ as well.

In order to compute $E(|X - Y|)$, we first need $P\{|X - Y| = k\}$. But this is not hard to compute: $|X - Y| = k$ means that either $X = 1$ and $Y = k + 1$ or $Y = 1$ and $X = k + 1$. Therefore, for all integers $k \geqslant 1$,

$$P\{|X - Y| = k\} = P\{Y = k + 1\} + P\{X = k + 1\} = p^k(1 - p) + (1 - p)^k p.$$

And therefore,

$$E(|X - Y|) = (1-p)\sum_{k=1}^{\infty} kp^k + p\sum_{k=1}^{\infty} k(1-p)^k.$$

One can in fact evaluate the sums: For all $0 < r < 1$ define

$$g(r) := \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots . \tag{1}$$

We know that $g(r) = (1-r)^{-1}$, of course. So we can differentiate g in two different ways. First,

$$g'(r) = 1 + 2r + 3r^2 + \cdots = \sum_{k=1}^{\infty} kr^{k-1}.$$

But also,

$$g'(r) = \frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{1}{(1-r)^2}.$$

Therefore,

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2}.$$

Or equivalently,

$$\sum_{k=1}^{\infty} kr^{k} = \frac{r}{(1-r)^2}.$$

Use this, once with $r := p$ and once with $r := 1-p$, to find that

$$E(|X - Y|) = (1-p)\left\{\frac{p}{(1-p)^2}\right\} + p\left\{\frac{1-p}{p^2}\right\} = \frac{p}{1-p} + \frac{1-p}{p}.$$

Problem 5, p. 302. X is binomial with parameters $n = 5$ and $p = 1/2$; and $Y = 5 - X$. Therefore, the table $f(x, y) = P\{X = x, Y = y\}$ is given as follows:

y/x	0	1	2	3	4	5
0	0	0	0	0	0	$(1/2)^5$
1	0	0	0	0	$5(1/2)^5$	0
2	0	0	0	$\binom{5}{3}(1/2)^5$	0	0
3	0	0	$\binom{5}{2}(1/2)^5$	0	0	0
4	0	$5(1/2)^5$	0	0	0	0
5	$(1/2)^5$	0	0	0	0	0