Partial Solutions to Homework 3
Mathematics 5010–1, Summer 2009

Problem 9, p. 181. We have

$$p(n+k) = \binom{n+k-1}{k-1}\left(1-\frac{\lambda}{k}\right)^k\left(\frac{\lambda}{k}\right)^n \quad \text{for } n \geqslant 0.$$

The combinatorial term is

$$\binom{k-1+n}{k-1} = \frac{(k-1+n)\cdots(k+1)k}{n!} \sim \frac{k^n}{n!} \quad \text{as } k \to \infty.$$

Because $(1-\frac{\lambda}{k})^k \sim e^{-\lambda}$ as $k \to \infty$, this shows us that

$$p(n+k) \sim \frac{k^n}{n!}e^{-\lambda}\frac{\lambda^n}{k^n} = \frac{e^{-\lambda}\lambda^n}{n!} \quad \text{as } k \to \infty.$$

Problem 14, p. 182. We need to compute $P\{X = x\}$. Note that there are $\binom{49}{6}$ possible [equally likely] outcomes in this experiment. In order to select a maximum of $x \in \{6, \ldots, 49\}$, we need to draw x [one way to do this] and from the remaining smaller $x-1$ numbers draw the remaining 5. Therefore, there are $\binom{x-1}{5} \times 1 = \binom{x-1}{5}$ ways such that we can get a maximum of x. Therefore, the probability $p(x)$ is $\binom{x-1}{5}/\binom{49}{6}$ for all $x = 6, \ldots, 49$; $p(x) = 0$ for other values of x.

Problem 25, p. 183. You will not need this problem to study for midterm #2.

Problem 1, pp. 198. The triangular random variable has mass function [see page 194]:

$$p(x) = \begin{cases} (n-x)/n^2 & \text{if } x = 1, \ldots, n, \\ (n+x)/n^2 & \text{if } x = -n, \ldots, 0. \end{cases}$$

(Or in compact form, $p(x) = (n-|x|)/n^2$ for $x = -n, \ldots, n$.) If $a = -n, \ldots, 0$, then a change of variables [$j := n + x$] shows that

$$F(a) = \sum_{x=-n}^{a} \frac{n+x}{n^2} = \sum_{j=0}^{n+a} \frac{j}{n^2} = \frac{(n+a)(n+a+1)}{2n^2}.$$

I have used the fact that $\sum_{j=1}^{b} j = b(b+1)/2$ for all $b \geqslant 1$.

Because $F(n) = 1$ it remains to compute $F(a)$ for $a = 1, \ldots, n-1$. In that case,

$$F(a) = 1 - P\{X \geqslant a+1\} = 1 - \sum_{x=a+1}^{n} \frac{n-x}{n^2}.$$

Change variables [$j := n - x$] once more to find that

$$F(a) = 1 - \sum_{j=0}^{n-a-1} \frac{j}{n^2} = 1 - \frac{(n-a-1)(n-a)}{2n^2}$$

Problem 2, pp. 198. Consider the general discrete probability mass function: $p(x_j) = p_j$ where $x_1, \ldots, x_n$ are distinct numbers, and p_j's are positive numbers such that $p_1 + \cdots + p_n = 1$.

Define $\Omega := \{x_1, \ldots, x_n\}$; the individual outcomes $x_1, \ldots, x_n$ have respective probabilities $p_1, \ldots, p_n$. Now define the random variable X, as a function on Ω, as follows: $X(x_j) = x_j$ for all $1 \leqslant j \leqslant n$. Then, $P\{X = x_j\} = P(\{x_j\}) = p_j$.

Problem 2, pp. 212. This is a simple problem: Just plug into the definition of expectations to find that

$$EX = \sum_{x=1}^{n} xp(x) = \frac{2}{n(n+1)} \sum_{x=1}^{n} x^2 = \frac{2}{n(n+1)} \left(1 + 4 + \cdots + n^2\right).$$

If you know the identity $\sum_{x=1}^{n} x^2 = n(n+1)(2n+1)/6$, then you can simplify this further [but this is not necessary]:

$$EX = \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3}.$$