Partial Solutions to Homework 2
Mathematics 5010–1, Summer 2009

Problem 2, p. 89. Here, Ω is the total collection of all possible 2-card hands; therefore, $\#\Omega = \binom{52}{2} = \frac{52\times 51}{2} = 1326$. In order to get a 21, you have to have a court card and an ace; there are 16 court cards and 4 aces. Because there are $4 \times 16 = 64$ pairs that sum to 21, the probability of getting a sum of 21 is $64/1326 = 32/663$.

Problem 13, p. 89. I am told this problem doesn't show up on some of your texts (strange!). Here is the problem, for your convenience: *Let* A *and* B *be events with* $P(A) = \frac{3}{5}$ *and* $P(B) = \frac{1}{2}$. *Show that* $\frac{1}{10} \leqslant P(A \cap B) \leqslant \frac{1}{2}$, *and give examples to show that both extremes are possible. Can you find bounds for* $P(A \cup B)$*?*

Because $A \cap B \subset B$, it follows that $P(A \cap B) \leqslant P(B) = \frac{1}{2}$. This is the desired upper bound. For the lower bound note that

$$\frac{3}{5} = P(A) = P(A \cap B) + P(A \cap B^c) \leqslant P(A \cap B) + P(B^c) = P(A \cap B) + \frac{1}{2}.$$

Solve to obtain $P(A \cap B) \geqslant \frac{1}{10}$.

Note that $\frac{3}{5} > \frac{1}{2}$. Therefore, choose any event A with $P(A) = \frac{3}{5}$, and any event $B \subset A$ with $P(B) = \frac{1}{2}$ to see that $P(A \cap B) = P(B) = \frac{1}{2}$.

- Here is an example of how this can happen: You choose an integer at random $\Omega = \{1, \ldots, 10\}$, all choices being equally likely. Let $B = \{1, 3, 5, 7, 9\}$ be the event that you choose an odd number, and $A := \{1, 3, 5, 7, 9, 10\}$. Then, $P(A) = \frac{3}{5}$, $P(B) = \frac{1}{2}$, and $B \subset A$ so $P(A \cap B) = P(B) = \frac{1}{2}$.

For the other bound, suppose $B^c \subset A$. Then, $P(A \cap B^c) = P(B^c) = \frac{1}{2}$ and $P(A \cap B) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$.

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Finally, in order to find bounds on $P(A \cup B)$ we recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Use the bounds for $P(A \cap B)$ to obtain

$$P(A \cup B) \leqslant P(A) + P(B) - \frac{1}{10} = \frac{3}{5} + \frac{1}{2} - \frac{1}{10} = 1;$$

and

$$P(A \cup B) \geqslant P(A) + P(B) - \frac{1}{2} = \frac{3}{5} + \frac{1}{2} - \frac{1}{2} = \frac{3}{5}.$$

Problem 20, p. 89. The answer to (a) is easy: $\alpha = a/(a+b)$; let us concentrate on the answer to (b).

Let A_j denote the event that the jth ball drawn is amber. We know that $P(A_1) = \alpha = a/(a+b)$. Let us make some calculations:

$$P(A_2 \mid A_1) = \frac{a+c}{a+b+c} \quad \text{and} \quad P(A_2 \mid A_1^c) = \frac{a}{a+b+c}.$$

Therefore,

$$\begin{aligned} P(A_2) &= P(A_2 \mid A_1)P(A_1) + P(A_2 \mid A_1^c)P(A_1^c) \\ &= \frac{a+c}{a+b+c} \cdot \frac{a}{a+b} + \frac{a}{a+b+c} \cdot \frac{b}{a+b} \\ &= \frac{(a+c)a + ab}{(a+b+c)(a+b)} \\ &= \frac{a(a+c+b)}{(a+b+c)(a+b)} = \frac{a}{a+b}. \end{aligned}$$

In other words, $P(A_2) = \alpha$ also.

Problem 23, p. 89. Let B_j denote the event that we have selected the card with j bees on it. [We have B_0, B_1, and B_2 to contend with.] Let $\mathcal{B}$ denote the event that the side is showing a bee. The problem asks for $P(B_1 \mid \mathcal{B})$.

Let us make some calculations:

$$P(\mathcal{B} \mid B_0) = 0; \; P(\mathcal{B} \mid B_1) = \frac{1}{2}; \; P(\mathcal{B} \mid B_2) = 1.$$

Also, $P(B_0) = P(B_1) = P(B_2) = \frac{1}{3}$. By Bayes' rule,

$$\begin{aligned} P(B_1 \mid \mathcal{B}) &= \frac{P(\mathcal{B} \mid B_1)P(B_1)}{P(\mathcal{B} \mid B_0)P(B_0) + P(\mathcal{B} \mid B_1)P(B_1) + P(\mathcal{B} \mid B_2)P(B_2)} \\ &= \frac{\frac{1}{2} \times \frac{1}{3}}{0 + \left(\frac{1}{2} \times \frac{1}{3}\right) + \frac{1}{3}} = \frac{1}{3}. \end{aligned}$$