Partial Solutions to Homework 1
Mathematics 5010–1, Summer 2009

Problem 1, p. 126. (a) There are $12^4 = 20,736$ different ways of assigning a zodiac sign to 4 people; $12 \times 11 \times 10 \times 9 = 11,880$ of them correspond to assignments in which no two people have the same sign. Since all possible outcomes are equally likely,

$$\mathrm{P}\{\text{all zodiac signs different}\} = \frac{12 \times 11 \times 10 \times 9}{12^4} \approx 0.572917.$$

(b) Suppose we have N people, where N is an integer $\leqslant 11$. Note that

$$\mathrm{P}\{\text{at least 2 have the same sign}\} = 1 - \mathrm{P}\{\text{none have the same sign}\}.$$

Therefore, we need only to compute

$$\mathrm{P}\{\text{none have the same sign}\} = \frac{12 \times 11 \times \cdots \times (13 - \mathrm{N})}{12^{\mathrm{N}}}.$$

Call this number $\mathrm{P}(\mathrm{N})$. Our goal is to find the smallest integer $\mathrm{N} \leqslant 11$ such that $1 - \mathrm{P}(\mathrm{N}) \geqslant 1/2$. Equivalently, find the smallest integer $\mathrm{N} \leqslant 11$ such that $\mathrm{P}(\mathrm{N}) \leqslant 1/2$. The following table makes the answer crystal clear:

N	**P(N)**
1	1
2	$\frac{11}{12} \approx 0.91667$
3	$\frac{11\times10}{12\times12} \approx 0.763889$
4	$\frac{11\times10\times9}{12\times12\times12} \approx 0.57291667$
5	$\frac{11\times10\times9\times8}{12\times12\times12\times12} \approx 0.3819445$

Therefore, the answer is $\mathrm{N} = 5$.

Problem 2, p. 126. Here, Ω is the collection of all possible ways of listing 5 digits. Each of the $10^5 = 100,000$ elements of Ω has the same chance of being selected. The number of elements that are comprised of all-different digits is $10 \times 9 \times 8 \times 7 \times 6$. Therefore,

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{10^5} = 0.3024 \approx 0.3.$$

Similarly, for 6 digits the answer is

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{10^6} = 0.1512.$$

Problem 3, p. 126. Note that Ω is the collection of ways to write down 4-digit codes using 4 different digits, repetition permitted. Therefore, Ω has $\binom{10}{4}$ elements, all of whom are equally likely to be selected.

(a) There are seven possible runs [those that end with $3, 4, \ldots, 9$]. Therefore, the answer is $7/\binom{10}{4}$.

(b) Assuming that "greater than five" means "six or more," then there are $\binom{4}{4} = 1$ way of choosing 4 digits from $\{6, 7, 8, 9\}$ and therefore the answer is $1/\binom{10}{4}$. If "greater than five" means "five or more," then the answer is $\binom{5}{4}/\binom{10}{4} = 5/\binom{10}{4}$.

(c) The answer is one minus the probability that none include zero. There are $\binom{9}{4}$ ways of selecting 4 nonzero digits. Therefore, the probability is

$$1 - \frac{\binom{9}{4}}{\binom{10}{4}}.$$

(d) If we interpret "greater than 7" to mean "$\geqslant 7$," then the answer is one minus the probability that they are all 6 or less. Because there are 6 digits $\leqslant 6$, the probability is

$$1 - \frac{\binom{7}{4}}{\binom{10}{4}}.$$

If we interpret "greater than 7" to mean "$\geqslant 8$," then the probability is

$$1 - \frac{\binom{8}{4}}{\binom{10}{4}}.$$

(e) There are 5 odd digits in $\{0, \ldots, 9\}$. Therefore, the probability is

$$\frac{\binom{5}{4}}{\binom{10}{4}} = \frac{5}{\binom{10}{4}}.$$

Problem 7, p. 126. The idea is to count the same number in two different ways. Namely, we first observe that there are $\binom{2n}{n}$ ways of choosing n digits from $\{1, \ldots, 2n\}$. Here is another way to count the same number: The collection of all possible ways to choose n digits from $\{1, \ldots, 2n\}$ can be partitioned into $n+1$ groups. Group 1 is the collection of all possible ways to choose n digits from $\{1, \ldots, 2n\}$ such that 0 are even and n are odd. The number of ways to choose zero even digits from $\{1, \ldots, 2n\}$ is $\binom{n}{0}$ because there are exactly n even digits in $\{1, \ldots, 2n\}$. The number of ways to choose n odd digits from $\{1, \ldots, 2n\}$ is $\binom{n}{n}$. Therefore, the principle of counting tells us that group one has $\binom{n}{0} \cdot \binom{n}{n}$ elements.

Similarly, let group 2 be the collection of all possible ways to choose n digits from $\{1, \ldots, 2n\}$ such that 1 is even and $n-1$ are odd. Group two has $\binom{n}{1} \cdot \binom{n}{n-1}$ elements in it.

In general, let group k denote the collection of all possible ways to choose n digits from $\{1, \ldots, 2n\}$ such that k are even and $n-k$ are odd [$k = 0, \ldots, n$]. Group k has $\binom{n}{k} \cdot \binom{n}{n-k}$.

Since Groups 0–n are disjoint and their totality is all possible ways of getting n digits from $\{1, \ldots, 2n\}$, it follows that the total number of ways of getting n digits from $\{1, \ldots, 2n\}$ is the same as the sum total of the number of elements of group 0 plus group 1 plus ... group n. Because this sum must be $\binom{2n}{n}$, we have shown that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}.$$

The result follows, because it is easy to check directly that $\binom{n}{k} = \binom{n}{n-k}$ for all $k = 0, \ldots, n$.

Problem 18, p. 128. Let p denote the probability that a flip of the coin yields heads; according to the chapter's notation, $q := 1-p$ is the probability that the coin yields tails on a toss. Then the probability that the first n are all heads and the next n are all tails is $p^n q^n = (pq)^n$. In fact, given any arrangement of n heads and n tails, the chances are $(pq)^n$ that the said arrangement shows up. Since no two distinct arrangements can occur at the same time, the probability that we can one of them is the sum. But all such arrangements have the same probability, namely $(pq)^n$. Therefore, the probability is

$$\text{total number of arrangements of } n \text{ heads and } n \text{ tails} \times (pq)^n.$$

The result follows because the number of ways to arrange exactly n heads [and therefore n tails] is $\binom{2n}{n}$ [select n spots for the heads, among the $2n$ possible spots].