Solutions to Assignment #7
Math 501–1, Spring 2006
University of Utah

Problems:

1. *Suppose Y is uniformly distributed on $(0\,,5)$. What is the probability that the roots of the equation $4x^2+4xY+Y+2=0$ are both real?*

Solution: The two roots of the quadratic are:

$$x=\frac{-4\pm\sqrt{16Y^2-16(Y+2)}}{8}=\frac{1}{2}\left[-1\pm\sqrt{Y^2-Y-2}\right].$$

The roots are real if and only if $Y^2-Y-2\geq 0$. Consider next the quadratic equation $y^2-y-2=0$. The solutions are

$$y=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm 3}{2}=-1\text{ or }2.$$

This means that $y^2-y-2=(y+1)(y-2)$, which can be check directly too. Consequently, $Y^2-Y-2\geq 0$ if and only if $(Y+1)(Y-2)\geq 0$. Because $0\leq Y\leq 5$, this means that $Y^2-Y-2\geq 0$ if and only if $Y-2\geq 0$. Thus, the probability of real roots is $P\{Y\geq 2\}=3/5$.

2. *Two fair dice are rolled. Find the joint mass function of $(X\,,Y)$ when:*

(a) *X is the maximum (i.e., largest) of the values of the two dice, and Y is the sum of the values of the two dice;*

Solution: The possible values of $(X\,,Y)$ are $(1\,,2)$, $(2\,,3)$, $(2\,,4)$, $(3\,,4)$, $\ldots$, $(3\,,6)$, $(4\,,5)$, $\ldots$, $(4\,,8)$, $(5\,,6)$, $\ldots$, $(5\,,10)$, $(6\,,7)$, $\ldots$, $(6\,,12)$. The probabilities are:
$p(1\,,2)=1/36$ (one and one);
$p(2\,,3)=2/36$ (a two and a one);
$p(3\,,4)=p(3\,,5)=2/36$, and $p(3\,,6)=1/36$;
$p(4\,,5)=p(4\,,6)=p(4\,,7)=2/36$ and $p(4\,,8)=1/36$;
$p(5\,,6)=p(5\,,7)=p(5\,,8)=p(5\,,9)=2/36$ and $p(5\,,5)=1/36$;
$p(6\,,7)=p(6\,,8)=p(6\,,9)=p(6\,,10)=p(6\,,11)=2/36$ and $p(6\,,12)=1/36$.

(b) *X is the value of the first die and Y is the maximum of the values of the two dice;*

Solution: This is done similarly to the previous one. The probabilities are:
$p(1\,,1)=1/36$; $p(2\,,2)=2/36$; $p(3\,,3)=3/36$; $\ldots$; $p(6\,,6)=6/36$;
$p(1\,,2)=p(1\,,3)=p(1\,,4)=p(1\,,5)=p(1\,,6)=1/36$;
$p(2\,,3)=p(2\,,4)=p(2\,,5)=p(2\,,6)=1/36$;
$p(3\,,4)=p(3\,,5)=p(3\,,6)=1/36$;
$p(4\,,5)=p(4\,,6)=1/36$;
$p(5\,,6)=1/36$;

(c) *X is the minimum (i.e., smallest) of the values of the two dice, and Y is the maximum of the two values.*

Solution: This is done similarly to the previous one. The probabilities are:
$p(1,1) = 1/36$ and $p(1,2) = p(1,3) = p(1,4) = p(1,5) = p(1,6) = 2/36$;
$p(2,2) = 1/36$ and $p(2,3) = p(2,4) = p(2,5) = p(2,6) = 2/36$;
$p(3,3) = 1/36$ and $p(3,4) = p(3,5) = p(3,6) = 2/36$;
$p(4,4) = 1/36$ and $p(4,5) = p(4,6) = 2/36$;
$p(5,5) = 1/36$ and $p(5,6) = 2/36$;
$p(6,6) = 1/36$.

3. *Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 denote the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of (X_1, X_2).*

Solution: The possible values are all two-dimensional integers of the form (i,j), were $i, j \geq 0$. Thus, we have

$$p(i,j) = P\left(F_1 \cap \cdots \cap F_i \cap S_{i+1} \cap F_{i+2} \cap \cdots \cap F_{j+2+j}\right) = p(1-p)^{i+j}, \quad i, j = 0, 1, 2, \ldots,$$

where $S_i := \{\text{success at the } i\text{th}\}$ and $F_i := S_i^c$.

4. *The joint density function of (X, Y) is given by*

$$f(x,y) = \begin{cases} c(y^2 - x^2)e^{-y}, & \text{if } -y \leq x \leq y \text{ and } 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) *Find c.*

Solution: Solve for c as usual:

$$\begin{aligned} 1 &= c \int_0^\infty \int_{-y}^{y} (y^2 - x^2) e^{-y}\, dx\, dy \\ &= c \int_0^\infty e^{-y} \left(\int_{-y}^{y} (y^2 - x^2)\, dx \right) dy \\ &= c \int_0^\infty e^{-y} \left(2y^3 - \frac{1}{3} x^3 \Big|_{-y}^{y} \right) dy \\ &= \frac{4c}{3} \int_0^\infty y^3 e^{-y}\, dy = \frac{4c}{3} \Gamma(4) = \frac{4c}{3} \times 3! = 8c. \end{aligned}$$

Therefore, $c = 1/8$.

(b) *Find the (marginal) density functions of X and Y respectively.*

Solution: Integrate each variable separately: First, suppose $x > 0$ and note that $f_X(x) = (1/8)\int_x^\infty (y^2 - x^2)e^{-y}\, dy$. Split the integrals and compute by parts to find that $\int_x^\infty y^2 e^{-y}\, dy = x^2 e^{-x} + 2\int_x^\infty y e^{-y}\, dy$. Also, $\int_x^\infty e^{-y}\, dy = e^{-x}$. Therefore,

$$f_X(x) = \frac{1}{4} \int_x^\infty y e^{-y}\, dy = \frac{1}{4}(x+1)e^{-x}, \qquad \text{for } x > 0.$$

If $x < 0$, then $f_X(x) = f_X(-x)$, by symmetry.

Next we compute f_Y: For all $y > 0$,

$$f_Y(y) = \frac{1}{8}\int_{-y}^{y}(y^2 - x^2)e^{-y}\,dx = \frac{1}{8}\left[2y^3e^{-y} - e^{-y}\int_{-y}^{y} x^2\,dx\right]$$
$$= \frac{1}{8}\left[2y^3e^{-y} - \frac{2}{3}y^3e^{-y}\right] = \frac{1}{6}y^3e^{-y}.$$

If $y < 0$ then $f_Y(y) = 0$.

(c) *Find $E(X)$.*

Solution: Because f_X is symmetric, $EX = 0$.

(d) Find $P\{X > Y\}$.

Solution: Zero because $P\{X > Y\} = \int\int_{x>y} f(x\,,y)\,dx\,dy$, and $f(x\,,y) =$ if $x > y$.

5. *The (joint) density function of $(X\,,Y)$ is given by*

$$f(x\,,y) = \begin{cases} e^{-(x+y)}, & \text{if } 0 \le x < \infty, \text{ and } 0 \le y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Find: **(a)** $P\{X < Y\}$*; and* **(b)** $P\{X < a\}$ *for all real numbers a.*

Solution: First of all, note that

$$f(x\,,y) = f_X(x)\cdot f_Y(y),$$

where

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y}, & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, X and Y are independent; both are exponentially distributed with mean one. In particular, $P\{X < Y\} = P\{X > Y\}$. Since $P\{X = Y\} = 0$, it follows then that $P\{X > Y\} = 1/2$. [Do this by integration as well!] On the other hand,

$$P\{X < a\} = \int_0^a e^{-x}\,dx = 1 - e^{-a}.$$

if $a > 0$, and $P\{X < a\} = 0$ if $a < 0$.