Exam 1, Math 5010-1

Fall 2016
September 28, 2016

> This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours.
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> The exam is made of 4 questions in 5 pages, and is worth 30 points, total. Be sure to try all of the problems.
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> Partial credit is given only to carefully-written solutions.

1. (10 points total) A fair, six-sided die is cast twice.

   (a) (5 points) Let X denote the total number of dots rolled. What is the probability mass function of X? (Hint: Start by identifying the possible values of X.)

   Solution. The possible values are all integers between 2 and 12. The probabilities are:

x	$\mathrm{P}\{X = x\}$
2	$^1/_{36}$
3	$^2/_{36}$
4	$^3/_{36}$
5	$^4/_{36}$
6	$^5/_{36}$
7	$^6/_{36}$
8	$^5/_{36}$
9	$^4/_{36}$
10	$^3/_{36}$
11	$^2/_{36}$
12	$^1/_{36}$

(b) (5 points) Let Y denote the maximum of the number of dots rolled on the two die. What is the probability mass function of Y?

Solution. The possible values are all integers between 1 and 6. The probabilities are:

x	$\mathrm{P}\{X = x\}$
1	$^1/_{36}$
2	$^3/_{36}$
3	$^5/_{36}$
4	$^7/_{36}$
5	$^9/_{36}$
6	$^{11}/_{36}$

2. (5 points) An experimenter has two 2-sided coins: One of the coins (call it coin 1) tosses heads with probability p_1; the other (call it coin 2) tosses heads with probability p_2. The experimenter selects one of the two coins at random (with probability $1/2$ each), and then tosses it independently 10 times. What is the probability that all ten tosses yield heads?

Solution. Let C_i denote the event that coin i were selected. We are asked to find $\mathrm{P}(H_1 \cap \cdots \cap H_{10})$, where H_j denotes the event that the jth toss yielded heads. By Bayes' formula,

$$\begin{aligned}\mathrm{P}(H_1 \cap \cdots \cap H_{10}) &= \mathrm{P}(H_1 \cap \cdots \cap H_{10} \mid C_1)\mathrm{P}(C_1) + \mathrm{P}(H_1 \cap \cdots \cap H_{10} \mid C_2)\mathrm{P}(C_2) \\ &= \left[p_1^{10} \times \tfrac{1}{2}\right] + \left[p_2^{10} \times \tfrac{1}{2}\right] = \frac{p_1^{10} + p_2^{10}}{2}.\end{aligned}$$

3. (10 points total) We have two boxes: The first box contains 1 black and 1 blue marble; the second box contains 2 black and 1 blue marbles. We select a box at random (all equally likely); then from the selected box we sample a marble at random (all equally likely).

   (a) (5 points) What is the probability that we sampled a blue marble?

   Solution. Let B_j denote the event that box j is selected, and W the event that a blue marble is drawn. We know that $\mathrm{P}(W \mid B_1) = 1/2$ and $\mathrm{P}(W \mid B_2) = 1/3$. We also know that $\mathrm{P}(B_1) = \mathrm{P}(B_2) = 1/2$. Therefore, by Bayes' formula,

   $$\mathrm{P}(W) = \mathrm{P}(W \mid B_1)\mathrm{P}(B_1) + \mathrm{P}(W \mid B_2)\mathrm{P}(B_2) = \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{3} \times \tfrac{1}{2}\right) = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{5}{12}.$$

   (b) (5 points) If we sampled a blue marble, then what is the conditional probability that first box was sampled?

   Solution. Consequently,

   $$\mathrm{P}(B_1 \mid W) = \frac{\mathrm{P}(W \cap B_1)}{\mathrm{P}(W)} = \frac{\mathrm{P}(W \mid B_1)\mathrm{P}(B_1)}{\mathrm{P}(W)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

4. (5 points) Suppose that every day of the year, the temperature is one of “cold,” “mild,” and “hot” with equal probability, and that daily temperatures are independent from one another. What is the probability that there is at least one cold day in July? (There are 31 days in July.)

Solution. This is a Binomial$(365\,,{}^1\!/\!{}_3)$ problem, where success means “cold,” and failure means “not cold.” Let X denote the total number of successes. Then,

$$\mathrm{P}\{X \geq 1\} = 1 - \mathrm{P}\{X = 0\} = 1 - \left(\tfrac{2}{3}\right)^{31}.$$