Math 3220–001, Summer 2013, Exam 5

1. (10 points) Evaluate

$$\lim_{N\to\infty}\int_0^N\int_0^1 \sqrt{x}\, e^{-y/x}\, dx\, dy.$$

Solution. By Fubini, the integral is equal to

$$\begin{aligned}\lim_{N\to\infty}\int_0^1\int_0^N \sqrt{x}\, e^{-y/x}\, dy\, dx &= \lim_{N\to\infty}\int_0^1 x^{3/2}\int_0^{N/x} e^{-y}\, dy\, dx\\ &= \lim_{N\to\infty}\int_0^1 x^{3/2}\left(1-e^{-N/x}\right) dx\\ &= \frac{2}{5}-\lim_{N\to\infty}\int_0^1 x^{3/2}e^{-N/x}\, dx.\end{aligned}$$

There are many different ways of seeing that the limit is zero. For example,

$$\int_0^1 x^{3/2}e^{-N/x}\, dx = N^{5/2}\int_{1/N}^{\infty} y^{3/2}e^{-y}\, dy, \qquad (y := x/N)$$

and this goes to zero as $N\to\infty$ by l'Hôpital's rule. Therefore, the preceding integral converges to zero as $N\to\infty$, and hence the answer is $2/5$.

2. (10 points) Let ϕ denote the 1-form $\phi(x\,,y) = x^2\, dx - dy$. Compute $\int_\gamma \phi$, where $\gamma : [0\,,1] \to \mathbb{R}^2$ denotes the curve $\gamma(t) := (\cos t\,, t)$.

Solution. $\phi_1(x\,,y) = x^2$, $\phi_2(x\,,y) = -1$, $\gamma'(t) = (-\sin t\,, 1)$. Therefore,

$$\begin{aligned}\int_\gamma \phi &= \int_0^1 \phi_1(\gamma(t))\gamma_1'(t)\, dt + \int_0^1 \phi_2(\gamma(t))\gamma_2'(t)\, dt\\ &= -\int_0^1 (\cos t)^2 \sin t\ dt - 1 = \int_1^{\cos 1} u^2\, du - 1 = \frac{(\cos 1)^3-1}{3}-1\\ &= -\frac{4-(\cos 1)^3}{3}.\end{aligned}$$

3. (10 points) Prove that a smooth parameter change does not change the length of a smooth curve.

Solution. Let $\gamma : [a\,,b] \to \mathbb{R}^p$ be a smooth curve, and $\alpha : [c\,,d] \to [a\,,b]$ a smooth parameter change from γ to $\lambda := \gamma \circ \alpha$. We have

$$\ell(\lambda) = \int_c^d \|\lambda'(t)\|\, dt = \int_c^d \left\|\gamma'(\alpha(t))\alpha'(t)\right\| dt,$$

thanks to chain rule. Now, $\alpha'(t)$ is a scalar and has a constant sign. Therefore, $\|\gamma'(t)\alpha'(t)\| = \alpha'(t)\|\gamma'(t)\|$. Therefore,

$$\ell(\lambda) = \int_c^d \|\gamma'(\alpha(t))\||\alpha'(t)|\, dt = \int_a^b \|\gamma'(s)\|\, ds,$$

after a change of variables. The latter quantity is $\ell(\gamma)$.

4. (10 points total) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates $(x\,,y) \in \mathbb{R}^2$ by $45°$; that is,

$$\varphi(x\,,y) = \begin{pmatrix} \dfrac{x+y}{\sqrt{2}} \\ \\ \dfrac{x-y}{\sqrt{2}} \end{pmatrix}.$$

Let $\mathcal{R}$ denote the rectangle $\mathcal{R} := [0\,,1] \times [0\,,1]$.

(a) (5 points) Plot $\varphi(\mathcal{R})$.

(b) (5 points) Compute, *using the change of variable formula of chapter 10*, the volume of $\varphi(\mathcal{R})$.

Solution. $\varphi(\mathcal{R})$ is the lozenge whose vertices are at $(0\,,0)$, $(1/\sqrt{2}\,,1/\sqrt{2})$, $(1/\sqrt{2}\,,-1/\sqrt{2})$, and $(\sqrt{2}\,,0)$. Note that φ is smooth, and

$$d\varphi(x\,,y) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \;\Rightarrow\; \det d\varphi(x\,,y) = -1.$$

We know, from Chapter 10, that

$$\begin{aligned} V(\varphi(\mathcal{R})) &= \int_{\mathcal{R}} |\det d\varphi(x\,,y)|\, dV(x\,,y) = \int_{\mathcal{R}} dV(x\,,y) \\ &= V(\mathcal{R}) = 1. \end{aligned}$$