Math 3220–001, Summer 2013, Solutions to Exam 3

1. *Consider the matrices*

$$\boldsymbol{A} := \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}, \quad \boldsymbol{B} := \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

   (a) *Compute* $\det \boldsymbol{A}$*.*
   Solution: $\det \boldsymbol{A} = (3 \times 1) - (-2 \times -2) = -1$.

   (b) Calculate $\boldsymbol{AB}$.
   Solution: $\boldsymbol{AB} = \begin{pmatrix} 4 & 4 \\ -3 & -3 \end{pmatrix}$.

   (c) Find $\boldsymbol{A}^{-1}$.
   Solution: $\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}$.

   (d) Find the rank of $\boldsymbol{B}$.
   Solution: $\text{rank}(\boldsymbol{B}) = 1$.

2. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *that is defined by* $f(x\,,y) := \sin(x)\cos(y)$ *for all* $x, y \in \mathbb{R}$*. Compute:*

   (a) $\dfrac{\partial f}{\partial x}$. **Solution:** $\cos(x)\cos(y)$.

   (b) $\dfrac{\partial f}{\partial y}$. **Solution:** $-\sin(x)\sin(y)$.

   (c) $\dfrac{\partial^2 f}{\partial x^2}$ **Solution:** $-\sin(x)\cos(y)$.

   (d) $\dfrac{\partial^2 f}{\partial x\,\partial y}$. **Solution:** $-\cos(x)\sin(y)$.

3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a continuously-differentiable function, and* $g : \mathbb{R} \to \mathbb{R}$ *is an affine function such that: (i)* $g(0) = f(0)$*; and (ii)*

$$\lim_{h \to 0} \frac{f(h) - g(h)}{h} = 0.$$

   Prove then that $g(x) = f(0) + f'(0)x$ *for all* $x \in \mathbb{R}$*.*

   Solution: Because g is affine, there exist constants a and b such that

$$g(x) = ax + b \qquad \text{for all } x \in \mathbb{R}.$$

Since $g(0) = f(0)$, it follows that $b = f(0)$; that is,

$$g(x) = ax + f(0) \qquad \text{for all } x \in \mathbb{R}.$$

But then

$$\frac{f(h) - g(h)}{h} = \frac{f(h) - [ah + f(0)]}{h} = \frac{f(h) - f(0)}{h} - a.$$

As $h \to 0$, the left-hand side converges to 0 by assumption, whereas the right-hand side converges to $f'(0) - a$ by the definition of a derivative. This shows that $0 = f'(0) - a$; equivalently, $a = f'(0)$, and hence

$$g(x) = f'(0)x + f(0) \qquad \text{for all } x \in \mathbb{R}.$$

4. *Compute the differential* $\mathrm{d}L$ *of* L*, where* $L : \mathbb{R}^p \to \mathbb{R}^q$ *is affine. [Hint: Start by writing* L*, in matrix form, as* $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$*.]*

   Solution: The hint says it all: Write $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ for all $\boldsymbol{x} \in \mathbb{R}^p$, where $\boldsymbol{b} \in \mathbb{R}^p$ is a vector and $\boldsymbol{A}$ is a $q \times p$ matrix. Then,

$$L(\boldsymbol{x} + \boldsymbol{h}) - L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{h} \qquad \text{for every } \boldsymbol{x}, \boldsymbol{h} \in \mathbb{R}^p.$$

   In particular,

$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{L(\boldsymbol{x} + \boldsymbol{h}) - L(\boldsymbol{x}) - \boldsymbol{A}\boldsymbol{h}}{\|\boldsymbol{h}\|} = \boldsymbol{0} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^p.$$

   This means that $\mathrm{d}L(\boldsymbol{x}) = \boldsymbol{A}$ for all $\boldsymbol{x} \in \mathbb{R}^p$.