The University of Utah, Spring 2002
Mathematics 3070–5, Midterm 1
Solutions

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1. *A truth serum has the property that 80% of the guilty suspects are properly judged. Moreover, innocent suspects are misjudged 2% of the time. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and if the serum indicates that he is guilty, what is the probability that he is innocent? Show your work.*

   Solution Let $G = \{\text{guilty}\}$, and $J = \{\text{judged guilty}\}$ to see that $G' = \{\text{innocent}\}$ and $J' = \{\text{judged innocent}\}$. Moreover, the problem tells us that $P\{G\} = 0.05$, $P\{J \,|\, G\} = 0.8$, and $P\{J \,|\, G'\} = 0.02$. We are after $P\{G' \,|\, J\}$.

   First, we compute $P\{J\}$: by Bayes' rule,

$$\begin{aligned} P\{J\} &= P\{J \cap G\} + P\{J \cap G'\} \\ &= P\{J \,|\, G\}P\{G\} + P\{J \,|\, G'\}P\{G'\} \\ &= 0.8 \times 0.05 + 0.02 \times 0.95 \\ &= 0.059. \end{aligned}$$

   Thus,

$$\begin{aligned} P\{G' \,|\, J\} &= P\{J \,|\, G'\} \cdot \frac{P\{G'\}}{P\{J\}} \\ &= 0.02 \times \frac{0.95}{0.059} \\ &\approx \boxed{0.322}. \end{aligned}$$

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2. *Three fair coins are tossed independently from one another. Let X denote the total number of heads, and Y denote the number of tails thus obtained.*

   (a) *Find the joint probability distribution of X and Y.*

   (b) *What is $P\{X \geq 2\}$?*

   Solution to (a): The joint distribution is given by the following.

		x 0	1	2	3
	0	0	0	0	$\frac{1}{8}$
y	1	0	0	$\frac{3}{8}$	0
	2	0	$\frac{3}{8}$	0	0
	3	$\frac{1}{8}$	0	0	0

   Solution to (b): $P\{X \geq 2\} = \frac{3}{8} + \frac{1}{8} = \boxed{\frac{1}{2}}$.

3. *Suppose the joint density function of X and Y is*

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) *Compute the marginal density functions of X and Y.*

(b) *Are X and Y independent? Explain clearly.*

Solution to (a): To find the marginal $g(x)$ of X, integrate $f(x,y)$ against dy, i.e.,

$$g(x) = \begin{cases} \displaystyle\int_0^1 4xy\,dy = 4x \int_0^1 y\,dy = 2x, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$h(y) = \begin{cases} 2y, & \text{if } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to (b): $\boxed{\text{Yes}}$, since $f(x,y) = g(x)h(y)$.

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4. *SAT verbal scores are known to be approximately distributed according to a normal curve. This year, 10,000 people took the test, and the average verbal SAT score was 550 points, and the standard deviation was 100 points. A graduate program in creative writing will only admit students whose SAT verbal score was in the top 2% of the applicant pool. What is the minimum score needed for admission in to this program? Show your work.*

Solution: The z-score for top 2% is between 2.05 and 2.06, which we take as $z = 2.055$, i.e.,

$$\frac{\mathsf{Min.score} - \mu}{\sigma} = 2.055.$$

Plug $\sigma = 100$ and $\mu = 550$ to get $\mathsf{Min.Score} = 2.055 \times 100 + 550 = \boxed{755.5}$ points.

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5. *Consider the random variables X and Y whose joint probability distribution is*

		x = 1	2	3
	1	0.05	0.05	0.1
y	2	0.05	0.1	0.35
	3	0	0.2	0.1

One can show that $E[X] = 2.45$, $E[Y] = 2.1$, $Var(X) \approx 0.5$, and $Var(Y) \approx 0.50$. Use this information to compute the correlation between X and Y.

Solution: We need the covariance, which is in turn found by first calculating $E[XY]$.

$$\begin{aligned} E[XY] &= (1 \times 1 \times 0.05) + (2 \times 1 \times 0.05) + (3 \times 1 \times 0.1) + (1 \times 2 \times 0.05) + (2 \times 2 \times 0.01) \\ &\quad + (3 \times 2 \times 0.35) + (1 \times 3 \times 0) + (2 \times 3 \times 0.2) + (3 \times 3 \times 0.1) = 4.79. \end{aligned}$$

So the covariance is

$$\sigma_{XY} = E[XY] - \mu_X \mu_Y = 4.79 - (2.45 \times 2.1) = -0.355.$$

Thus, the correlation is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \frac{-0.355}{\sqrt{0.5} \times \sqrt{0.5}} \approx \boxed{-0.71}.$$